

\mathcal{R} : Darboux curves on surfaces

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Abstract

In 1872 G. Darboux defined a family of curves on surfaces of \mathbb{R}^3 which are preserved by the action of the Möbius group and share many properties with geodesics. Here we characterize these curves under the view point of Lorentz geometry, prove some general properties and make them explicit on simple surfaces, retrieving in particular results of Pell (1900) and Santaló (1941).

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Introduction

Our interest is to understand the extrinsic conformal geometry of a surface $M \subset \mathbb{R}^3$ or $M \subset \mathbb{S}^3$, that is objects invariant by the action of the Möbius group. Here we study a family of curves on a surface called *Darboux curves*. They are characterized by a relation between the geometry of the curve and the surface: the osculating sphere to the curve is tangent to the surface. Almost as in the case of geodesics, through every point and direction which is not of principal curvature, passes a unique Darboux curve. References about these curves are [Da1], [Ha], [Ri], [Co], [Pe], [En], [Sa1], [Sa2], [Se], [Ve].

The analogy with geodesics goes further: the osculating spheres along a Darboux curve are geodesics, that we call \tilde{D} -curves in the set $V(M)$ of spheres tangent to the surface and having with M a saddle-type contact, endowed with a natural semi-riemannian metric.

We will first consider the foliations by lines of curvature, and the foliations \mathcal{F}_α , defined by the line field making a constant angle α with the first foliation by lines of curvature, as they will provide reference foliations to understand the behavior of Darboux curves; on the way, transverse properties of these foliations \mathcal{F}_α will be obtained for isothermic surfaces.

The dynamics of Darboux curves is obtained from the study of \tilde{D} -curves, which almost define a flow on $V(M)$. **R: articulate both studies** We will study the angle drift of Darboux curves with respect to the foliations of the surface by principal curvature lines.

In particular we study the Darboux curves on some particular canal surfaces and on quadrics.

*Steven Verpoort called the attention of the authors on the fact that in algebraic geometry **R: check** the term “Darboux curves”, has a different meaning. As the name of Darboux is meaningful for many readers from differential geometry or dynamical systems, the authors finally decided to maintain the name “Darboux curves” here.*

1 Preliminaries

1.1 The set of spheres in \mathbb{S}^3

The Lorentz quadratic form \mathcal{L} on \mathbb{R}^5 and the associated Lorentz bilinear form $\mathcal{L}(\cdot, \cdot)$, are defined by $\mathcal{L}(x_0, \dots, x_4) = -x_0^2 + (x_1^2 + \dots + x_4^2)$ and $\mathcal{L}(u, v) = -u_0v_0 + (u_1v_1 + \dots + u_4v_4)$.

The Euclidean space \mathbb{R}^5 equipped with this pseudo-inner product \mathcal{L} is called the *Lorentz space* and denoted by \mathbb{L}^5 .

The isotropy cone $\mathcal{L}i = \{v \in \mathbb{R}^5 \mid \mathcal{L}(v) = 0\}$ of \mathcal{L} is called the *light cone*. Its non-zero vectors are also called *light-like* vectors. The *light-cone* divides the set of vectors $v \in \mathbb{L}^5, v \notin \{\mathcal{L} = 0\}$ in two classes:

A vector v in \mathbb{R}^5 is called *space-like* if $\mathcal{L}(v) > 0$ and *time-like* if $\mathcal{L}(v) < 0$.

A straight line is called space-like (or time-like) if it contains a space-like (or respectively, time-like) vector.

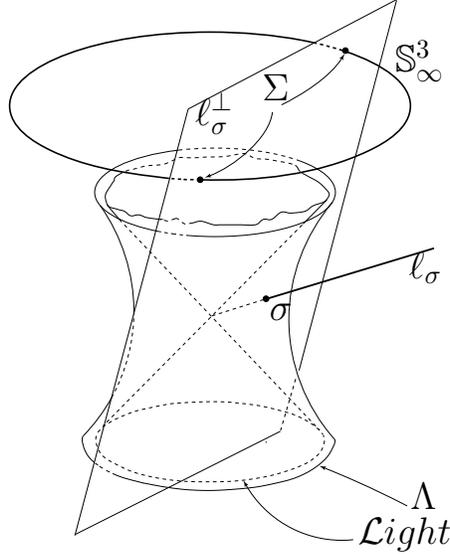


Figure 1: \mathbb{S}_∞^3 and the correspondence between points of Λ^4 and spheres.

The points at infinity of the light cone in the upper half space $\{x_0 > 0\}$ form a 3-dimensional sphere. Let it be denoted by \mathbb{S}_∞^3 . Since it can be considered as the set of lines through the origin in the light cone, it is identified with the intersection \mathbb{S}_1^3 of the upper half light cone and the hyperplane $\{x_0 = 1\}$, which is given by $\mathbb{S}_1^3 = \{(x_1, \dots, x_4) \mid x_1^2 + \dots + x_4^2 - 1 = 0\}$.

To each point $\sigma \in \Lambda^4 = \{v \in \mathbb{R}^5 \mid \mathcal{L}(v) = 1\}$ corresponds a sphere $\Sigma = \sigma^\perp \cap \mathbb{S}_\infty^3$ or $\Sigma = \sigma^\perp \cap \mathbb{S}_1^3$ (see Figure 1). Instead of finding the points of \mathbb{S}^3 “at infinity”, we can also consider the section of the light-cone by a space-like affine hyperplane H_z tangent to the upper sheet of the hyperboloid $\mathcal{H} = \{\mathcal{L} = -1\}$ at a point z . This intersection $\mathcal{L}ight \cap H_z$ inherits from the Lorentz metric a metric of constant curvature 1 (see [H-J], [La-Wa], and Figure 2).

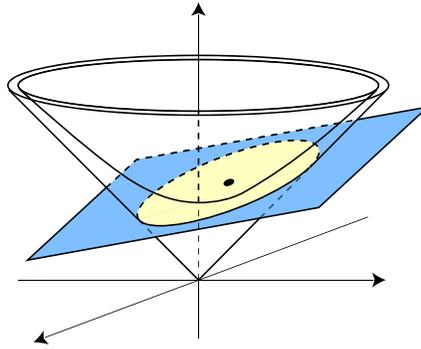


Figure 2: A tangent space to \mathbb{H}^5 cuts the light cone at a unit sphere

Notice that the intersection of Λ^4 with a space-like plane P containing the origin is a

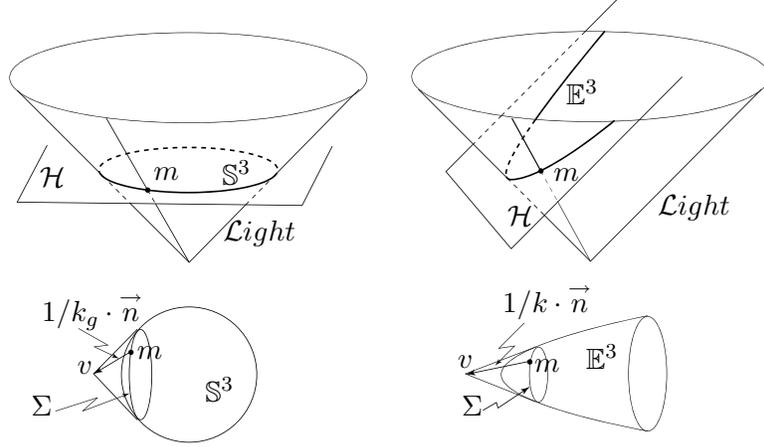


Figure 3: Spherical and Euclidean models in the Minkowski space \mathbb{L}^5 (up). The geodesic curvature k_g , picture in the affine hyperplane \mathcal{H} (down).

circle $\gamma \subset \Lambda^4$ of radius one in P . The points of this circle correspond to the spheres of a pencil with base circle. The arc-length of a segment contained in γ is equal to the angle between the spheres corresponding to the extremities of the arc.

It is convenient to have a formula giving the point $\sigma \in \Lambda^4$ in terms of the Riemannian geometry of the corresponding sphere $\Sigma \subset \mathbb{S}^3 \subset \mathcal{L}ight$ and a point m on it. For that we need to know also the unit vector \vec{n} tangent to \mathbb{S}^3 and normal to Σ at m and the geodesic curvature of Σ , that is the geodesic curvature k_g of any geodesic circle on Σ .

Proposition 1. *The point $\sigma \in \Lambda^4$ corresponding to the sphere $\Sigma \subset \mathbb{S}^3 \subset \mathcal{L}ight$ is given by*

$$\sigma = k_g m + \vec{n}. \quad (1)$$

Remark: A similar proposition can be stated for spheres in the Euclidean space \mathbb{E}^3 seen as a section of the light cone by an affine hyperplane parallel to an hyperplane tangent to the light cone.

The proof of Proposition 1 can be found in [H-J] and [La-Oh]. The idea of the proof is shown on Figure 3: Let \mathcal{H} be the affine hyperplane such that $\mathbb{S}^3 = \mathcal{L}ight \cap \mathcal{H}$, let P be the hyperplane such that $\Sigma = \mathbb{S}^3 \cap P$; the vertex of the cone, contained in \mathcal{H} , tangent to \mathbb{S}^3 along Σ is a point of the line P^\perp which contains the point $\sigma \in \Lambda^4$.

1.2 Curves in Λ^4 and canal surfaces

A differentiable curve $\gamma = \gamma(t)$ is called *space-like* if, at each point its tangent vector $\dot{\gamma}(t)$ is space-like, that is $\mathcal{L}(\dot{\gamma}) > 0$; it is called *time-like* if, at each point its tangent vector $\dot{\gamma}(t)$ is time-like, that is $\mathcal{L}(\dot{\gamma}) < 0$ (see Figure 5). When the curve is time-like, the spheres are nested. When the curve is space-like, the family of spheres Σ_t associated to the points $\gamma(t)$ defines an envelope which is a surface, union of circles called the *characteristic circles* of the surface. From now on we will suppose that the space-like curve γ is parameterized

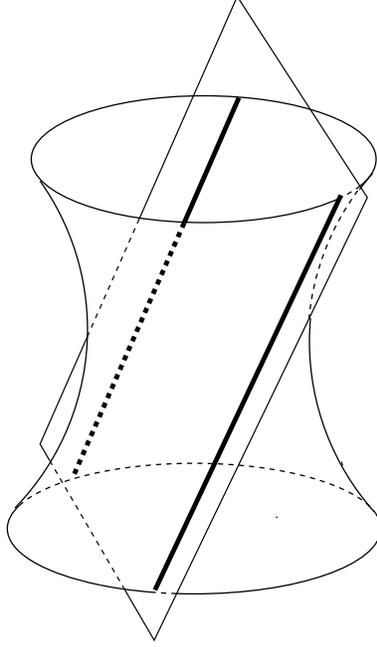


Figure 4: Lightrays in Λ^4 corresponding to a pencil of tangent spheres

by arc-length, that is $|\mathcal{L}(\dot{\gamma})| = 1$. There is one characteristic circle Γ_{Car} on each sphere Σ_t of the family and it is the intersection of Σ_t and the sphere $\dot{\Sigma}_t = [\text{Span}(\dot{\gamma}(t))]^\perp \cap \mathbb{S}^3$.

We call such an envelope of one-parameter family of spheres a *canal surface*. The sphere $\Sigma(t)$ is tangent to the canal surface along the characteristic circle except at maybe two points where the surface is singular. We say that the curve $\gamma(t)$ and the one-parameter family of spheres $\Sigma()$, the canal surface envelope of the family *correspond*. Reciprocally, when a one-parameter of spheres admits an envelope, the corresponding curve is space-like, as the existence of an envelope forces nearby spheres to intersect. One can refer, for example, to [La-Wa] for proofs concerning canal surfaces.

An extra condition is necessary to guarantee that the envelope is immersed. The *geodesic curvature vector* $\vec{k}_g = \ddot{\gamma}(t) + \gamma(t)$ should be time-like. We call the envelope of the spheres Σ_t corresponding to the points of such a curve γ a *regular canal surface*.

When the geodesic curvature vector is space-like, the three spheres $\Sigma(t) = (\gamma(t)^\perp \cap \mathbb{S}^3)$, $\dot{\Sigma}(t) = (\dot{\gamma}(t)^\perp \cap \mathbb{S}^3)$ and $\ddot{\Sigma}(t) = (\ddot{\gamma}(t)^\perp \cap \mathbb{S}^3)$ intersect in two points $m_1(t)$ and $m_2(t)$. These two points form locally two curves.

Lemma 2. *Let $\gamma = \{\gamma(t)\} \subset \Lambda^4$ be a space-like curve which has space-like geodesic curvature vector. Then the canal surface, envelope of the spheres $\Sigma(t) = (\gamma(t)^\perp \cap \mathbb{S}^3)$ has two cuspidal edges. The two points of the cuspidal edges belonging to the characteristic circle $Car(t)$ of the canal are $[\mathbb{R}\gamma(t) \oplus \mathbb{R}\dot{\gamma}(t) \oplus \mathbb{R}\ddot{\gamma}(t)]^\perp \cap \mathbb{S}^3$; the characteristic circles are tangent to the two cuspidal edges.*

Here, let us just prove that the characteristic circles are tangent to the curves $m_i(t)$.

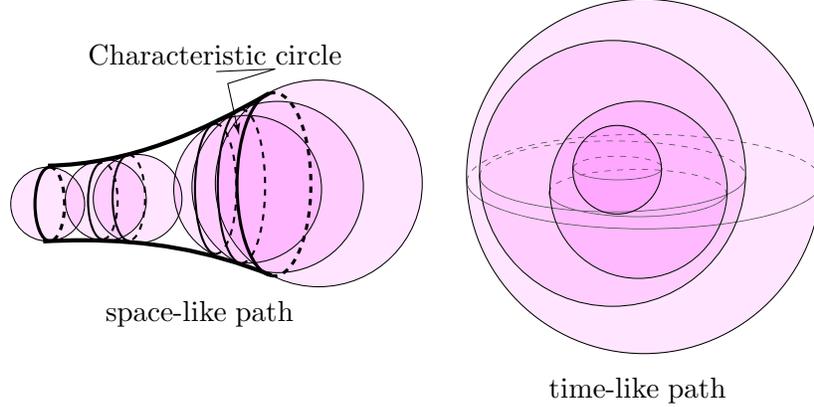


Figure 5: Spheres corresponding to space-like and time-like paths.

We need to prove that $\dot{m}_i(t)$ is tangent to $Car(t)$, that is tangent to $\Sigma(t)$ and to $\dot{\Sigma}(t)$. This is the case if $(\dot{m}_i(t), \gamma(t)) = (\dot{m}_i(t), \dot{\gamma}(t)) = 0$. As the point $m_i(t)$ belong to the three spheres $\Sigma(t)$, $\dot{\Sigma}(t)$ and $\ddot{\Sigma}(t)$, we now that $(m_i(t), \gamma(t)) = (m_i(t), \dot{\gamma}(t)) = (m_i(t), \ddot{\gamma}(t)) = 0$. Derivating the first two Lorentz scalar products, and using the previous equalities, we get the desired relations $(\dot{m}_i(t), \gamma(t)) = (\dot{m}_i(t), \dot{\gamma}(t)) = 0$.

Notice that when a regular point μ of the envelope tends to a point $m_i(t)$ of the singular locus, the tangent plane at μ tends to the tangent plane at $m_i(t)$ to the sphere $\Sigma(t)$.

When the geodesic curvature vector is light-like, we call the curve a *drill*. Then generically it is the family of osculating spheres to a curve $C \subset \mathbb{S}^3$ (see [Tho] and [La-So]). The characteristic circles are in that case the osculating circles of the curve.

2 Spheres and surfaces

2.1 Spheres tangent to a surface

From Proposition 1, we see that the points in Λ^4 corresponding to a pencil of spheres tangent to a surface M at a point m form two parallel light-rays (one for each choice of normal vector n). Let us now choose the normal vector n , and consider the spheres $\Sigma_{m,k}$ associated to the points $\sigma_{m,k} = km + n$. All the spheres $\Sigma_{m,k}$, but for at most two, have either a center contact or a saddle contact with M (see Figures 6 and 8).

When m is not an umbilic, the exceptional spheres correspond to the principal curvatures k_1 and k_2 (we suppose that $k_1 < k_2$) of M at m ; they are called osculating spheres.

We need to consider the germ at m of the intersection of a sphere tangent to M at m and M , that we will call *local intersection*. That is, we will not consider the whole intersection of a sphere tangent at m to M , but only the intersection $(\mathcal{U} \cap \Sigma \cap M)$ of $\Sigma \cap M$ with a small enough neighborhood $\mathcal{U} \subset \mathbb{S}^3$ of m .

When $k \notin [k_1, k_2]$, the local intersection of $\Sigma_{m,k}$ and M near the origin reduces to a point, the origin (*center contact*).

When $k \in]k_1, k_2[$, the local intersection of $\Sigma_{m,k}$ and M near the origin consists of two curves intersecting transversely at m (*saddle contact*).

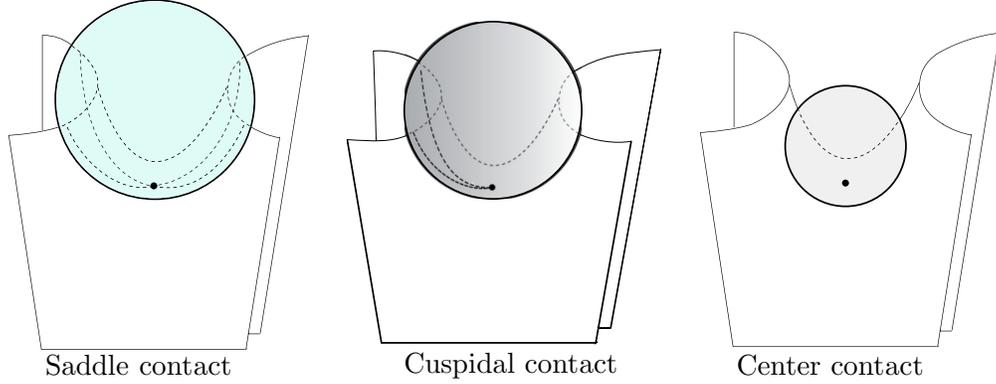


Figure 6: Possible contacts of a sphere and a surface

When $k = k_1$ or $k = k_2$, the local intersection of $\Sigma_{m,k}$ and M near the origin is a curve singular at m ; the singularity is in general of cuspidal type.

The points of Λ^4 corresponding to osculating spheres associated to k_1 form a surface \mathcal{O}_1 . We complete the surface with the osculating spheres at umbilics of M . In the same way we get a second surface \mathcal{O}_2 which intersect \mathcal{O}_1 only at the osculating spheres at umbilics.

When $k \in]k_1, k_2[$ goes to k_1 , the two tangents at the common point to the intersection $\Sigma_k \cap M$ tend to the principal direction associated to k_1 . This shows that the principal directions are conformally defined

Let us call \mathcal{F}_0 the foliation by lines of curvature associated to k_1 . It is also a conformal object, as the foliation by lines of curvature associated to k_2 .

We will use the direction tangent to the leaves of the foliation \mathcal{F}_0 as origin for angles in the set of lines in the planes tangent to M . Therefore the foliation associated to k_2 is $\mathcal{F}_{\pi/2} = \mathcal{F}_{-\pi/2}$. We denote by X_1 a field of unit vectors tangents to \mathcal{F}_0 , and by X_2 a field of unit vectors tangents to $\mathcal{F}_{\pi/2}$.

We can now also define a one-parameter family of foliations \mathcal{F}_α , $\alpha \in [-\pi/2, \pi/2]$ (or $\alpha \in \mathbb{P}_1$). \mathcal{F}_α is the foliation the leaves of which make a constant angle α with the leaves of \mathcal{F}_0 .

We will study these foliations in Section 5.

Definition 3. *The 3-manifold $V(M) \subset \Lambda^4$ is the set of spheres having a saddle contact with the surface $M \subset \mathbb{S}^3$.*

It is a submanifold of Λ^4 with boundary the union of the two surfaces \mathcal{O}_1 and \mathcal{O}_2 .

Euler's computation of the curvatures of sections of a surface by normal planes implies the

Proposition 4. *Let $k = k_1 \cos^2 \alpha + k_2 \sin^2 \alpha$. Then the angle of the tangents direction $\ell_{\pm\alpha}$ at m to $\Sigma_{m,k} \cap M$ with the principal direction corresponding to k_1 is $\pm\alpha$.*

When the reference to the angle α is useful, we will also use the notation $\Sigma_{m,\alpha}$ instead of $\Sigma_{m,k}$ for the sphere of curvature $k = k_1 \cos^2 \alpha + k_2 \sin^2 \alpha$ tangent to M at the point m .

In general, the intersection of an osculating sphere with M admits a cuspidal point at m , the tangent to the cusp is then the principal direction associated to the curvature k_i of the osculating sphere.

It is proved in [La-Oh] that the points of Λ^4 corresponding to the osculating spheres associated to k_1 along a line of curvature associated to k_1 , that is a leaf of \mathcal{F}_0 , form a light-like curve. Therefore \mathcal{F}_0 lifts to a foliation $\tilde{\mathcal{F}}_0$ of the surface \mathcal{O}_1 by light-like curves. Let us recall the proof here. It uses the formula $\sigma = km + n$ giving the point $\sigma \in \Lambda$ corresponding to a sphere $\Sigma \subset \mathbb{R}^3$ of curvature k and unit normal vector n (see Proposition 1). Derivating it with respect to arc-length on the line of curvature we get $\sigma' = k'm + km' + n' = k'm + kX_1 - k_1X_1$, as the sphere of curvature $k_1(m(s))$ is tangent to M along a line of curvature where $n' = -k_1X_1$.

Notice that a one-parameter family of spheres tangent to a curve can never be time-like, as a time-like curve gives locally nested spheres which cannot be tangent to a curve. Therefore a curve in \mathcal{O}_1 formed of osculating spheres to M along a curve C have a space-like tangent vector, except when the curve C is tangent to \mathcal{F}_0 . This implies that, at a regular point of \mathcal{O}_1 the restriction to $T\mathcal{O}_1$ of the Lorentz metric is degenerated. The kernel at a point σ , a sphere tangent to M at m , is the light direction $Span(m)$.

Definition 5. A point $m \in M$ is a ridge point for k_1 if m is a critical point for the restriction of k_1 to the line of curvature for k_1 through the point m .

The lift of a ridge point to \mathcal{O}_1 is in general a cusp of a leaf L of $\tilde{\mathcal{F}}_0$. To see that, let us parameterize a leaf L of \mathcal{F}_0 near a ridge point using a regular parameter on the corresponding line of curvature; as, at a ridge point, $k'_1 = 0$, we get at the lift of the ridge point, $\sigma' = 0$.

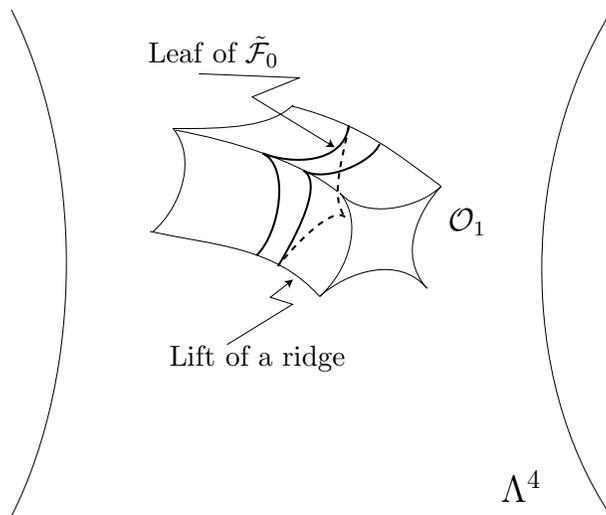


Figure 7: The surface \mathcal{O}_1 and the foliation $\tilde{\mathcal{F}}_0$

In general, ridge points form curves in M that we call just ridges. The lift to \mathcal{O}_1 of a ridge is in general a cuspidal edge of the surface \mathcal{O}_1 .

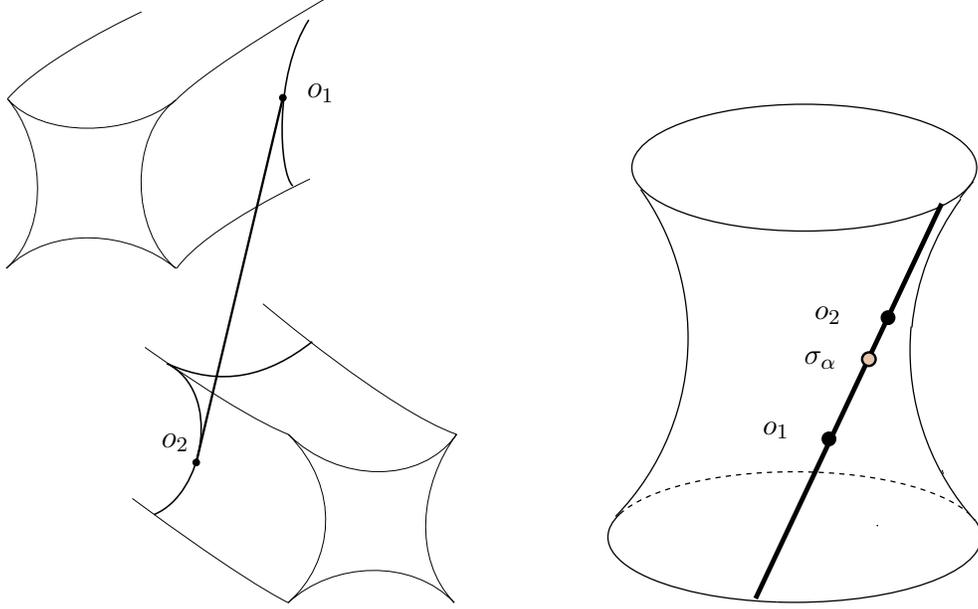


Figure 8: A light segment of $V(M)$ between two osculating spheres

Above each point $m \in M$ which is not an umbilic, the spheres tangent to the surface M having a saddle contact with M form an interval of boundary the two osculating spheres at m (see Figure 8).

Therefore, when M has no umbilical point, $V(M)$ is an interval fiber-bundle $\pi : V(M) \rightarrow M$ over M ; the boundary of $V(M)$ is the surface $\mathcal{O}_1 \cup \mathcal{O}_2$ of spheres osculating M . When M has umbilical points, the two folds \mathcal{O}_1 and \mathcal{O}_2 meet at osculating spheres at umbilical points of M .

Proposition 6. *At regular points, $V(M)$ inherits from Λ^4 a semi-Riemannian metric.*

Proof: Let us prove that, at each point $\sigma \in V(M)$, $T_\sigma V(M)$ is contained in $T_m \mathcal{L}ight = (\mathbb{R}m)^\perp$. For that, consider two curves in $V(M)$ of origin σ which project on two lines of curvature on M orthogonal at m . Suppose that their respective arc-lengths are u and v . Then, derivating $\sigma = km + n$ respectively with respect to u and v , we get

$$\sigma_u = (km + n)_u = k_u m + kX_1 - k_1 X_1$$

$$\sigma_v = (km + n)_v = k_v m + kX_2 - k_2 X_2$$

The vectors X_1 and X_2 are tangent to $M \subset \mathbb{S}^3 \subset \mathcal{L}ight$, they are therefore contained in $T_m \mathcal{L}ight = (\mathbb{R}m)^\perp$. The restriction of \mathcal{L} to $T_m \mathcal{L}ight$ is degenerated. The restriction of \mathcal{L} to any subspace of $T_m \mathcal{L}ight$ containing the light direction $\mathbb{R}m$, as $T_\sigma V(M)$, is therefore also degenerated.

Then the direction $\mathbb{R}m$, $m = \pi(\sigma)$ of the light ray through the point σ is the kernel direction of the restriction of the Lorentz metric to $T_\sigma V(M)$. The direction normal to tangent space $T_\sigma V(M)$ is also $(\mathbb{R}m)$. \square

The points σ_α of a fiber I_m can be expressed as linear combinations of the two osculating spheres $o_1(m)$ and $o_2(m)$: $\sigma_\alpha = \cos^2 \alpha o_1(m) + \sin^2 \alpha o_2(m)$. The sphere Σ_α corresponding to the point σ_α intersect the surface M in a neighborhood of m in two

arcs making an angle α with the first principal direction (corresponding to $o_1(m)$) (see Proposition 4).

The interval bundle $V(M)$ is closely related with the projective tangent bundle $\mathbb{P}T_1(M)$.

Let us chose as origin on a fiber of $\mathbb{P}T_1(M)$ the direction of the first principal direction. The “antipodal” direction of the first principal direction on the fiber P_m of $\mathbb{P}T_1(M)$ above m is the second principal direction. Sending the direction making an angle α with the first principal direction and the one making an angle $-\alpha$ to the point $\sigma_\alpha \in I_m$ “folds” the circle P_m on the interval I_m .

2.2 Spheres tangent to a surface along a curve

Let us consider now a curve $C \subset M$.

The restriction of $V(M)$ to C forms a two-dimensional surface in Λ^4 which is a light-ray interval bundle $V(C)$ over C out of the umbilical points of M which may belong to C . From $V(M)$, we get on $V(C)$ an induced semi-Riemannian metric.

In this text we will use $'$ for derivatives with respect to parameterization of curves contained in \mathbb{R}^3 or \mathbb{S}^3 , (often the parameter is an arc length), and \bullet for derivatives with respect to a parameterization of a curve in Λ^4 (often the parameter is an arc length, but now for the metric induced from the Lorentz “metric”).

Definition 7. We denote by $\Sigma_{m,\ell}$ the sphere tangent to the surface M at the point m such than one branch of the intersection $\Sigma_{\dot{c}} \cap M$ is tangent to the direction ℓ (canonical sphere associated to the direction ℓ). We will also use the notation $\Sigma_{m,v}$ when the direction ℓ is generated by a non-zero vector v .

Remark: We can rephrase Proposition 4, saying that, if ℓ_α is the line of $T_m M$ making at m the angle α with $T_{\mathcal{F}_0}$, then $\Sigma_{m,k} = \Sigma_{m,\ell_\alpha}$ when $k = k_1 \cos^2 \alpha + k_2 \sin^2 \alpha$.

Given a curve $C \subset M$ such that the tangent vector to C at $c(t)$ is contained in ℓ , $\Sigma_{c(t),\dot{c}(t)}$ is, among the spheres tangent to M at m , the one which have the best contact at m with the curve C .

At each point $m \in C$ such that the tangent to C is not a principal direction, there is a unique sphere $\Sigma_C(m)$ such that one branch of the local intersection $\Sigma_C(m) \cap M$ is tangent to C at m . If the tangent to C at m is a principal direction, we take $\Sigma_C(m) = o_i(m)$, the osculating sphere of the surface corresponding to the principal direction. We call the family of spheres $\Sigma_C(m)$ the *canonical family along C* , and denote it by $CanSec(C)$; the envelope $CanCan(C)$ of the spheres $\Sigma_C(m) \in CanSec(C)$ is called *the canonical canal* corresponding to $C \subset M$.

The point $CanSec(C)(m) \in V(C)$ of the *canonical section* of $V(C)$ corresponds to this sphere $\Sigma_C(m)$.

Proposition 8. The canonical section $CanSec(C)$ of $V(C)$ satisfies the following properties:

- i) The geodesic curvature vector of the curve $CanSec(C) \subset V(M)$ satisfies $\vec{k}_g \in T_m V(M)$, and therefore is space-like.
- ii) The section $CanSec(C)$ is a geodesic in $V(C)$.
- iii) $CanSec(C)(m) = k_n \cdot m + n$, where k_n is the normal curvature of M at $m \in M$ in the direction of the vector $C'(m)$; n is as usual the normal vector to $M \in \mathbb{S}^3$ at m .

In the proof, we will need a lemma from [La-So].

Lemma 9. *A curve $\Gamma = \{\gamma(t)\} \subset \mathbb{S}^3$ has contact of order $\geq k$ with a sphere Σ corresponding to σ iff*

$$\sigma \perp \text{span}(\gamma(t), \dot{\gamma}(t), \dots, \gamma^{(k)}(t))$$

Proof. The sphere Σ is the zero level of the function $f(x) = \langle x | \sigma \rangle$. Then the contact of $\Gamma(t)$ and Σ has the order of the zero of $(f \circ \gamma)(t) = \langle \gamma(t) | \sigma \rangle$. \square

Proof. (of the proposition) The condition defining the sphere $\Sigma_C(m)$ implies that the order of contact of C and $\Sigma_C(m)$ is at least 2, one more than the order of contact of $\Sigma_C(m) \cap M$ and C , which is at least one. To verify this property, notice that, in terms of the arc-length of a branch of $\Sigma_C(m) \cap M$, or equivalently of the arc-length s on C , the angle of $\Sigma(m)$ with M along $\Sigma_C(m) \cap M$ is of the order of s , if not smaller. The distance to C is of the order of s^2 , if not smaller. Therefore the order of the distance of a point of C to $\Sigma_C(m)$ is of order s^3 if not smaller. This means that C and the sphere $\Sigma_C(m)$ have contact of order at least 2 at m .

As the sphere Σ_C has contact order at least 2 with the curve C , $\text{CanSec}(C)(m)$ is orthogonal to m , \dot{m} and \ddot{m} .

To avoid heavy notation we will now denote by σ a point of the section $\text{CanSec}(C)$ parameterized by arc-length in Λ^4 .

Differentiating the relation $\langle \sigma(m) | \dot{m} \rangle = 0$ and using the relation $\langle \sigma(m) | \ddot{m} \rangle = 0$ we get $\langle \dot{\sigma}(m) | \dot{m} \rangle = 0$. Differentiating the relation $\langle \dot{\sigma}(m) | m \rangle = 0$, and using the relation $\langle \dot{\sigma}(m) | \dot{m} \rangle = 0$, we get $\langle \ddot{\sigma}(m) | m \rangle = 0$.

Recall that (see Subsection 1.2), the geodesic curvature vector $\vec{k}_g(m)$ is the orthogonal projection of $\ddot{\sigma}(m)$ on $T_{\sigma(m)}\Lambda^4$. One has $\vec{k}_g(m) = \ddot{\sigma}(m) + \sigma(m)$. It is therefore orthogonal to the line $\mathbb{R}m$, and therefore belongs to $(\mathbb{R}m)^\perp = T_{\sigma(m)}V(M)$, proving item i) of the proposition. As the tangent space to $V(M)$ is everywhere light-like, any direction different from the light-direction is space-like.

The geodesic curvature vector \vec{k}_g is orthogonal to $\mathbb{R}m$ and to $\dot{\sigma}(m)$, it is therefore orthogonal to $T_{\sigma(m)}V(C) = \mathbb{R}m \oplus \mathbb{R}\dot{\sigma}(m)$, proving that $\text{CanSec}(C)$ is a geodesic in $V(C)$, proving item ii).

Item iii) is just the formula of Proposition 1, as the sphere such that one branch of the local intersection with M at m is tangent to C at m has curvature k_n . \square

In order to compute explicitly the vector \vec{k}_g and to show Proposition 10, we will use the Darboux frame T, N_1, n, m of the curve $C \subset M \subset \mathbb{S}^3 \subset \text{Light}$, where T is the unit tangent vector to C , N_1 is the unit vector tangent to M normal to C compatible with the orientation of M and n the unit vector normal to M and tangent to \mathbb{S}^3 .

Proposition 10. *The section $\text{CanSec}(C)$ is the shortest of the sections of $V(C)$.*

Using again the formula $\sigma = k_n m + n$, where k_n is the normal curvature of M in the direction tangent to C at m , we see that, when σ is the section $\text{CanSec}(C)$

$$|\sigma'| = |k'_n m + k_n m' + n'| = |k'_n m - \tau_g N_1| = |\tau_g|, \quad (2)$$

where the *geodesic torsion* τ_g is defined by the following formula

$$\tau_g = -\langle \nabla_T n | N_1 \rangle. \quad (3)$$

Observe that our formula (2) gives an interpretation of the geodesic torsion of a curve $C \subset M$ as the rotation speed of the canonical family of spheres tangent to M along C .

In order to compute the geodesic curvature vector of the canonical section $CanSec(C)$, we need to use its parameterization by arc-length in Λ^4 . Then $\dot{\sigma} = \sigma' \frac{1}{\tau_g} = \frac{1}{\tau_g} (k'_n m - \tau_g N_1)$.

Differentiating once more, we get

$$\ddot{\sigma} = -\frac{\dot{\tau}_g}{\tau_g^3} (k'_n m - \tau_g N_1) + \frac{1}{\tau_g^2} [k''_n m + k'_n T - \tau'_g N_1 - \tau_g (-k_g T + \tau_g n)].$$

Notice that k_g is the geodesic curvature of $C \subset M$, while \vec{k}_g is the geodesic curvature vector of the curve $CanSec(C) \subset \Lambda^4$.

Simplifying, we get

$$\ddot{\sigma} = \phi(s)m + \frac{1}{\tau_g^2} [(k'_n + \tau_g k_g)T - \tau_g^2 n]$$

As $\vec{k}_g = \ddot{\sigma} + \sigma$ we get

$$\vec{k}_g = \psi(s)m + \frac{1}{\tau_g^2} (k'_n + \tau_g k_g)T \quad (4)$$

As the formula shows that the vector \vec{k}_g is orthogonal to m and to $\dot{\sigma} = \frac{1}{\tau_g} (k'_n m - \tau_g N_1)$, we verify that the curve $CanSec(C)$ is a geodesic on $V(C)$.

For another family of spheres tangent to M along C , in particular for another section of $V(C)$, where $\sigma = k m + n$, we see that spheres tangent to a surface along a curve form a space-like curve in Λ^4 ; explicitly we get

$$|\dot{\sigma}| = \left| \frac{1}{\tau_g} [(k - k_n)'m + (k - k_n)T + \tau_g N_1] \right|. \quad (5)$$

Since m , T and N are mutually orthogonal in \mathbb{L}^5 , this proves the fact that the section $CanSec(C)$ has minimal length among sections. Formula (5) shows also that no other section of $V(C)$ is of critical length.

Remark: $\tau_g ds$ is the differential of the rotation of the sphere $\Sigma_{c(t), \dot{c}(t)}$ (see Definition 7) along the curve C .

In fact, the sphere $\Sigma_{c(t), \dot{c}(t)}$ has a tangent movement which is a rotation of “axis” the characteristic circle of the family, which is tangent to C . The plane tangent to M along C is tangent to $\Sigma_{c(t), \dot{c}(t)}$ and therefore its tangent movement is a rotation of axis tangent to the previous characteristic circle which is tangent to C .

We see that changing the sphere tangent to the surface along C changes the “pitch” term but not the “roll” term (see Figure 2.2). The characteristic circles of the canal corresponding to a non-canonical section are not in general tangent to C .

The characteristic circle of the envelope of the family $\Sigma_{c(t), \dot{c}(t)}$ is the intersection of $span(\sigma, \dot{\sigma})^\perp$ and the sphere $\mathbb{S}^3 \subset \mathbb{R}^4$; as the vector T is orthogonal to m , n and N_1 , and

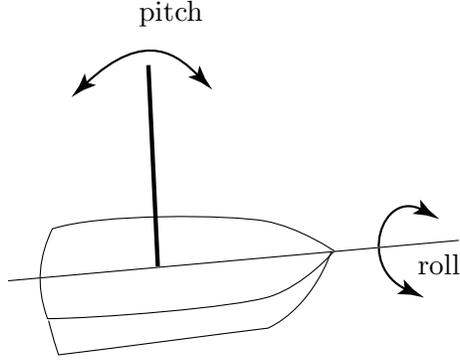


Figure 9: Pitch and roll

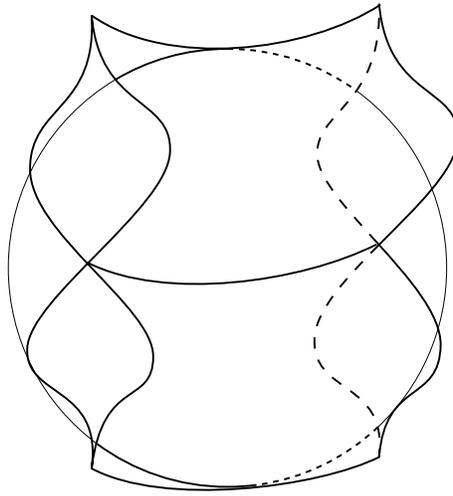


Figure 10: A canal with (locally) two cuspidal edges which is the envelope associated to a space-like curve with space-like geodesic curvature

therefore to σ and σ' (and also $\dot{\sigma}$), the characteristic circle is tangent at m to C . Other families of spheres will have their characteristic circles transverse to C ; in particular if the spheres are the planes tangent to M along C the characteristic lines of the family of planes are transverse to C when C is never tangent to an asymptotic direction. When C is tangent to a principal direction, the characteristic line of the family of tangent planes is orthogonal to C .

Proposition 11. *Let C be a curve contained in the surface $M \subset \mathbb{S}^3$. Suppose that it is nowhere tangent to a principal direction of curvature. Then the curve C is one fold of the singular locus of the canal surface $CanCan(C)$ defined by $CanSec(C)$.*

Proof: We already proved (Lemma 2) that the curve $CanSec(C)$ is space-like with a space-like geodesic acceleration at every point. The envelope $CanCan(C)$ is therefore a singular canal with (locally) two cuspidal edges (see Figure 10). The condition defining

the canonical sphere $\Sigma_{c(t), \dot{c}(t)}$ and Remark 2.2 guarantees that the characteristic circle of the family $\Sigma_{c(t), \dot{c}(t)}$ is tangent to C , which is therefore a singular curve of the canal. \square

2.3 Local conformal invariants of surfaces

Assume that S is a surface which is *umbilic free*, that is, that the principal curvatures $k_1(x)$ and $k_2(x)$ of S are different at any point x of S . Let X_1 and X_2 be unit vector fields tangent to the curvature lines corresponding to, respectively, k_1 and k_2 . Throughout the paper, we assume that $k_1 > k_2$. Put $\mu = (k_1 - k_2)/2$. Since more than 100 years, it is known ([Tr], see also [CSW]) that the vector fields $\xi_i = X_i/\mu$ and the coefficients θ_i ($i = 1, 2$) in

$$[\xi_1, \xi_2] = -\frac{1}{2}(\theta_2\xi_1 + \theta_1\xi_2) \quad (6)$$

are invariant under arbitrary (orientation preserving) conformal transformation of \mathbb{R}^3 . (In fact, they are invariant under arbitrary conformal change of the Riemannian metric on the ambient space. This follows from the known (see [La-Wa], page 142, for instance) relation $\tilde{A} = e^{-\phi}(A - g(\nabla\phi, N) \times \text{Id})$ between the Weingarten operators of a surface S with respect to conformally equivalent Riemannian metrics $\tilde{g} = e^{2\phi}g$ on the ambient space; here $\nabla\phi$ and N denote, respectively, the g -gradient of ϕ and the g -unit normal to S .) Elementary calculation involving Codazzi equations shows that

$$\theta_1 = \frac{1}{\mu^2} \cdot X_1(k_1) \quad \text{and} \quad \theta_2 = \frac{1}{\mu^2} \cdot X_2(k_2).$$

The quantities θ_i ($i = 1, 2$) are called *conformal principal curvatures* of S .

Let ω_1, ω_2 be the 1-forms dual to the vectors ξ_1, ξ_2 .

Lemma 12. *Consider the notation above. We have that*

$$d\omega_1 = \frac{1}{2}\theta_2\omega_1 \wedge \omega_2, \quad d\omega_2 = \frac{1}{2}\theta_1\omega_1 \wedge \omega_2 \quad (7)$$

Proof:

$$\begin{aligned} d\omega_1(\xi_1, \xi_2) &= \xi_1\omega_1(\xi_2) - \xi_2\omega_1(\xi_1) - \omega_1[\xi_1, \xi_2] \\ &= \frac{1}{2}\omega_1(\theta_2\xi_1 + \theta_1\xi_2) = \frac{1}{2}\theta_2 \end{aligned}$$

The second equality is proved in the same way. \square

3 Characterization in $V(M)$ and equations of Darboux curves

The osculating sphere o_C to a curve C at a point m is the unique sphere containing the osculating circle to the curve at the point which has a contact with the curve of order larger than the contact of the osculating circle with the curve. This shows that the osculating sphere is a conformally defined object.

After these preliminary remarks, the following definition of Darboux curves becomes more natural.

Definition 13. - Darboux curves on a surface $M \subset \mathbb{R}^3$ or $M \subset \mathbb{S}^3$ are the curves such that everywhere the osculating sphere is tangent to the surface.

- Darboux curves in $V(M)$ are families of osculating spheres to Darboux curve in M .

Let us insist: the definition of Darboux curves involves only spheres and contact order, so this notion belongs to conformal geometry.

We will now show that the osculating spheres along a Darboux curve form geodesics in $V(M)$.

Recall that a *drill* (see Subsection 1.2) is a curve in the space of spheres the geodesic curvature of which is light-like at each point. Generically, points of a drill are osculating spheres to the curve $C \in \mathbb{S}^3$ defined by the geodesic acceleration vector \vec{k}_g of the drill.

We see that if we can find drills in $V(m)$ we find geodesics. In fact we find that way almost all of them.

3.1 A geometric relation satisfied by Darboux curves

Theorem 14. A curve C is a Darboux curve if and only if the section $CanSec(C) \subset V(M)$ is a geodesic in $V(M)$. This happens if and only if the curve $C \subset M$ satisfies the equation:

$$k'_n + \tau_g k_g = 0. \quad (8)$$

We will prove the theorem after a remark and a proposition.

Remark: The light rays of $V(M)$ are also geodesics. Segments of geodesics of $V(M)$ which are not tangent to light rays define an arc of curve $C \subset M$, and therefore, when \vec{k}_g is light-like, C is a piece of Darboux curve on M . Also when C is a Darboux curve the only singular curve of the envelope $CanCan(C)$ is C . See [Tho].

Proof:(of Theorem 14)

Let Σ be a sphere tangent to M at m (with saddle contact), and σ be the corresponding point of Λ^4 . We have seen that the direction normal to $T_\sigma V(M)$ is $\mathbb{R}m$. Thomsen ([Tho], see also [La-So]) proved that the osculating spheres to a curve form a curve $\gamma \subset \Lambda^4$ with light-like geodesic acceleration vectors. Moreover the geodesic acceleration at a point (a sphere osculating the curve at m) is on the light-ray $\mathbb{R}m$. Conversely, a curve γ in Λ^4 with geodesic acceleration light-like everywhere provides a curve $C \subset \mathbb{S}^3$ or $C \subset \mathbb{R}^3$ such that the osculating spheres correspond to the points of γ .

Therefore Darboux curves in $V(M)$ have their geodesic acceleration proportional to m . As the normal to $V(M)$ at a sphere σ tangent to m is the line $\sigma + \mathbb{R}m$, these curves are geodesics of $V(M)$. Reciprocally a geodesic $\sigma(t)$ of $V(M)$ should have its geodesic acceleration (as curve of Λ^4) orthogonal to $T_\sigma V(M)$ everywhere. This means that when σ is tangent to M at m , this geodesic acceleration is proportional to m .

A Darboux curve $\mathcal{D} \subset V(M)$ is a particular case of canonical section of spheres tangent to M along a curve D . Therefore it satisfies equation $4 \vec{k}_g = \psi(s)m + \frac{1}{\tau_g^2}(k'_n + \tau_g k_g)T$. In order to be a geodesic in $V(M)$, the geodesic curvature vector of \mathcal{D} is a multiple of m ; this is the case if and only if the curve D satisfies the equation $(k'_n + \tau_g k_g) = 0$ \square

A Darboux curve $D \subset M$ have a better contact with the intersections of the spheres $\Sigma_{\mathcal{D}} \cap M$ than “ordinary” curves $C \subset M$ with the intersection $\Sigma_C \cap M$.

Proposition 15. Equation (8) in Theorem 14 is equivalent to the fact that one branch of the intersection of the sphere $\Sigma_{c(t), \dot{c}}$ has at the point the same geodesic curvature as the curve C .

Proof: When the direction ℓ defined by $\dot{c}(t)$ is not a principal direction, there is a unique sphere tangent to M which has contact with C of best order, the others having contact of order one with C . There is only one such sphere, in the pencil containing the osculating circle to C which is tangent to M . This last sphere should therefore be $\Sigma_{c(t), \dot{c}(t)}$

As the sphere $\Sigma_{c(t), \dot{c}}$ is osculating the curve C when it is a Darboux curve, the contact order with the curve should be at least 3, that is C and one branch of $\Sigma \cap M$ should have the same geodesic curvature. Otherwise, one should consider the sphere Σ_0 tangent to M at $c(0)$. As the tangent plane to M turns with speed τ_g along the intersection $\Sigma_0 \cap M$, the vertical distance between $C(t) \in C$ and σ , estimated starting from a point of $\bar{c}(t) \Sigma_0 \cap M$ close to $c(t)$ is of order $t \cdot (d(\bar{c}(t), c(t)))$. If the geodesic curvature of C and the tangent branch of $\Sigma_0 \cap M$ at $c(0)$ were different this order would be that of t^3 , too large for the sphere Σ_0 to osculate the curve C at $c(0)$. \square

3.2 Differential Equation of Darboux curves in a principal chart

R: Maybe, one can get a proof using Λ^4 and the fact geodesic curvature in $V(M)$ of a \tilde{D} -curve is zero. We should express the curve using two light-like curves and α .

We have seen that, in $V(M)$, the \tilde{D} -curves are geodesics and form almost a flow: two curves go through every point of the interior of $V(M)$. To get a flow, we should “unfold” the intervals of light-ray fibering $V(M)$ into circles, obtaining a flow on $\mathbb{P}(TM)$. We keep, through the point m, α , the inverse image of the two \tilde{D} curves starting at the point σ_α which projects on the Darboux curve making the angle α with the first principal direction of curvature, that is with \mathcal{F}_0 (see Subsection 2.2). In fact we will consider, in order to compute using an angle $\alpha \in \mathbb{S}^1$, the double cover T^1M , unit tangent bundle of M , of $\mathbb{P}(TM)$.

Consider a local principal chart (u, v) in a surface $\mathbb{M} \subset \mathbb{R}^3$, that is a chart obtained taking two lines of curvature intersecting at $m_0 \in M$, as $v = 0$ and $u = 0$ axes, and imposing that leaves of \mathcal{F}_0 are given by the value of v and leaves of $\mathcal{F}_{\pi/2}$ are given by the value of u .

In this chart, the first fundamental form writes

$$I = Edu^2 + Fdv^2$$

(the “ F ” term is zero as the levels of u and v are orthogonal). The second fundamental forms writes

$$II = edu^2 + gdv^2$$

The principal curvatures are $k_1 = e/E$ and $k_2 = g/G$ (see Section 2.3). **R:** no, section 2.3 does not provide this information. See do Carmo?

Proposition 16. Let (u, v) be a principal chart and c be a curve parameterized by arc length s making an angle $\alpha(s)$ with the principal direction $\partial/\partial u$. The angle α that a

Darboux curve c make with \mathcal{F}_0 satisfies the following differential equation

$$\begin{aligned} 3(k_1 - k_2) \sin \alpha \cos \alpha \frac{d\alpha}{ds} &= \frac{1}{\sqrt{E}} \frac{\partial k_1}{\partial u} \cos^3 \alpha + \frac{1}{\sqrt{G}} \frac{\partial k_2}{\partial v} \sin^3 \alpha \\ &= \frac{\partial k_1}{\partial s_1} \cos^3 \alpha + \frac{\partial k_2}{\partial s_2} \sin^3 \alpha. \end{aligned} \quad (9)$$

R: what are s_1 and s_2 ? Arc-length on lines of curvature? Then they do not provide a chart... Are s_1 and s_2 used somewhere?

Here $(u', v') = \left(\frac{\cos \alpha}{\sqrt{E}}, \frac{\sin \alpha}{\sqrt{G}}\right)$.

Proof. Consider a principal chart (u, v) such that $v = \text{const.}$ are the leaves of the principal foliation \mathcal{P}_1 .

Let $c(s) = (u(s), v(s))$ be a regular curve parameterized by arc length s . So we can write $c'(s) = (u', v') = \left(\frac{\cos \alpha}{\sqrt{E}}, \frac{\sin \alpha}{\sqrt{G}}\right)$, defining a direction α with respect to the principal foliation \mathcal{P}_1 .

We have the following classical relations:

$$\begin{aligned} k_n(\alpha) &= k_1 \cos^2 \alpha + k_2 \sin^2 \alpha, \\ k_g &= \frac{d\alpha}{ds} + k_g^1 \cos \alpha + k_g^2 \sin \alpha, \\ \tau_g &= (k_2 - k_1) \cos \alpha \sin \alpha. \end{aligned}$$

where k_g^1 and k_g^2 are the geodesic curvatures of the coordinates curves.

R: Brutal, pelo menos, preciso de uma referencia

Therefore,

$$\begin{aligned} \frac{dk_n}{ds} &= \frac{1}{\sqrt{E}} \frac{\partial k_1}{\partial u} \cos^3 \alpha + \frac{1}{\sqrt{G}} \frac{\partial k_1}{\partial v} \cos^2 \alpha \sin \alpha + \frac{1}{\sqrt{E}} \frac{\partial k_2}{\partial u} \cos \alpha \sin^2 \alpha \\ &\quad + \frac{1}{\sqrt{G}} \frac{\partial k_2}{\partial v} \sin^3 \alpha + 2(k_2 - k_1) \cos \alpha \sin \alpha \frac{d\alpha}{ds} \end{aligned}$$

We have seen in Subsection 3.1 that the differential equation of Darboux curves is given by $k'_n + k_g \tau_g = 0$ and so it follows that:

$$\begin{aligned} [k_g^1(k_2 - k_1) + \frac{1}{\sqrt{G}} \frac{\partial k_1}{\partial v}] \cos^2 \alpha \sin \alpha + [k_g^2(k_2 - k_1) + \frac{1}{\sqrt{E}} \frac{\partial k_2}{\partial u}] \cos \alpha \sin^2 \alpha \\ + 3(k_2 - k_1) \cos \alpha \sin \alpha \frac{d\alpha}{ds} + \frac{1}{\sqrt{E}} \frac{\partial k_1}{\partial u} \cos^3 \alpha + \frac{1}{\sqrt{G}} \frac{\partial k_2}{\partial v} \sin^3 \alpha = 0 \end{aligned}$$

In any orthogonal chart ($F = 0$) we have that $G_u = 2G\sqrt{E}k_g^2$ and $E_v = -2E\sqrt{G}k_g^1$. Also the Codazzi equations in a principal chart are given by:

$$\frac{\partial k_1}{\partial v} = \frac{E_v}{2E}(k_2 - k_1), \quad \frac{\partial k_2}{\partial u} = \frac{G_u}{2G}(k_1 - k_2).$$

See for example Struik's book [St] pp. 113 and 120.

Therefore,

$$\begin{aligned}
k_g^1(k_2 - k_1) + \frac{1}{\sqrt{G}} \frac{\partial k_1}{\partial v} &= -\frac{E_v}{2E\sqrt{G}}(k_2 - k_1) + \frac{1}{\sqrt{G}} \frac{E_v}{2E}(k_2 - k_1) = 0 \\
k_g^2(k_2 - k_1) + \frac{1}{\sqrt{E}} \frac{\partial k_2}{\partial u} &= \frac{G_u}{2G\sqrt{E}}(k_2 - k_1) + \frac{1}{\sqrt{E}} \frac{G_u}{2G}(k_1 - k_2) = 0.
\end{aligned}$$

This ends the proof. \square

Remark: A curve $c(s)$ has contact of third order with the associated osculating sphere, tangent to the surface, when

$$\langle c', c' \rangle [2\langle N', c'' \rangle + \langle N'', c' \rangle] - 3\langle c', N' \rangle \langle c', c'' \rangle = 0.$$

This equation can be used to obtain the differential equation of Darboux curves in any chart (u, v) . See [Sa1].

Remark: Let $\theta(s)$ be the angle between the unit normal N of the surface M and the principal normal n of a curve $c(s)$ parameterized by arc length s . Then c is a Darboux curve on M if and only if $k' \cos \theta + k\tau \sin \theta = 0$, where k and τ are the curvature and torsion of c . Just observe that $k_n = k \cos \theta$, $k_g = k \sin \theta$ and $\tau_g = \tau + \theta'$. Direct substitution in the equation $k'_n + k_g \tau_g = 0$ leads to the result. See also [Ve]. We can give to Equation (9) a conformally invariant form, using the conformal curvatures defined in Subsection 2.3.

$$\sin \alpha \cos \alpha \frac{d\alpha}{ds} = \frac{1}{12}(k_1 - k_2)[\theta_1 \cos^3 \alpha + \theta_2 \sin^3 \alpha].$$

Here $\theta_1 = 4\frac{1}{\sqrt{E}}\frac{\partial k_1}{\partial u}/(k_1 - k_2)^2$ and $\theta_2 = 4\frac{1}{\sqrt{G}}\frac{\partial k_2}{\partial v}/(k_1 - k_2)^2$ are the conformal principal curvatures.

R: More about the invariant viewpoint?

4 A plane-field on $V(M)$

In this section we will consider a natural plane field associated to the Darboux curves and the condition of integrability, in terms of conformal invariants, will be established. Also a connection with isothermic surfaces will be developed.

The two tangents to the two Darboux curves through the point $(m, \alpha) \in V(M)$ define a plane in $T_{m,\alpha}V(M)$. The ensemble of these planes defines a plane-field \mathcal{D} . It will be called *Darboux plane field*. The next proposition it will be useful to give an explicit parameterization of $V(M)$.

Proposition 17. *Consider a principal chart (u, v) and let $m(u, v)$ be a parameterization of $M \subset \mathbb{S}^3$; we denote by $N \in T_m\mathbb{S}^3$ the normal vector at m to M . The set of spheres $V(M) \subset \Lambda^4$ is parametrized by*

$$\varphi(u, v, \alpha) = k_n(\alpha)m(u, v) + N(u, v) \tag{10}$$

with $k_n(\alpha) = k_1(u, v) \cos^2 \alpha + k_2(u, v) \sin^2 \alpha$, $\mathcal{L}(m, m) = 0$, $\mathcal{L}(m, N) = 0$, and $N_u = -k_1 m_u$, $N_v = -k_2 m_v$.

Proof. Consider a principal chart (u, v) ; the parameterization φ sends the osculating sphere with radius $1/k_n(\alpha)$, tangent to M at the point $m(u, v)$ to a point in Λ^4 . We have that

$$\begin{aligned}\varphi_u &= \left[\frac{\partial k_1}{\partial u} \cos^2 \alpha + \frac{\partial k_2}{\partial u} \sin^2 \alpha \right] m + (k_n - k_1) m_u \\ \varphi_v &= \left[\frac{\partial k_1}{\partial v} \cos^2 \alpha + \frac{\partial k_2}{\partial v} \sin^2 \alpha \right] m + (k_n - k_2) m_v \\ \varphi_\alpha &= [(k_2 - k_1) \cos \alpha \sin \alpha] m\end{aligned}$$

So $D\phi$ has rank 3 for $\alpha \in (0, \frac{\pi}{2})$. \square

Proposition 18. *Consider a principal chart (u, v) . The Darboux plane field is defined locally by the vector fields*

$$\begin{aligned}\mathcal{D}_1^c &= \xi_1 + \frac{1}{6} \theta_1 \frac{\cos \alpha}{\sin \alpha} \frac{\partial}{\partial \alpha} \\ \mathcal{D}_2^c &= \xi_2 + \frac{1}{6} \theta_2 \frac{\sin \alpha}{\cos \alpha} \frac{\partial}{\partial \alpha}.\end{aligned}$$

Also it is defined by the differential 1-form

$$\Omega = \cos^2 \alpha \theta_1 \omega_1 + \sin^2 \alpha \theta_2 \omega_2 - 6 \sin \alpha \cos \alpha d\alpha.$$

Here ω_1 and ω_2 are dual to the conformal vector fields ξ_1 and ξ_2 and are given by: $\omega_1 = \sqrt{E} \frac{k_1 - k_2}{2} du$, $\omega_2 = \sqrt{G} \frac{k_1 - k_2}{2} dv$.

Proof. The first part follows from the differential equation (9) of Darboux curves in a principal chart.

In fact, consider in the unitary tangent bundle the suspension of the Darboux curves. These curves are defined by the following vector field

$$\mathcal{D}_1 = \frac{\cos \alpha}{\sqrt{E}} \frac{\partial}{\partial u} + \frac{\sin \alpha}{\sqrt{G}} \frac{\partial}{\partial v} + \left[\frac{\partial k_1 / \partial u}{3\sqrt{E}(k_1 - k_2)} \frac{\cos^2 \alpha}{\sin \alpha} + \frac{\partial k_2 / \partial v}{3\sqrt{G}(k_1 - k_2)} \frac{\sin^2 \alpha}{\cos \alpha} \right] \frac{\partial}{\partial \alpha}.$$

Consider the involution $\iota(u, v, \alpha) = (u, v, -\alpha)$ \mathcal{R} : Pawel replaced φ by ι and the induced vector field $\mathcal{D}_2 = \iota_*(\mathcal{D}_1)$. So,

$$\mathcal{D}_2 = \frac{\cos \alpha}{\sqrt{E}} \frac{\partial}{\partial u} - \frac{\sin \alpha}{\sqrt{G}} \frac{\partial}{\partial v} - \left[-\frac{\frac{\partial k_1}{\partial u}}{3\sqrt{E}(k_1 - k_2)} \frac{\cos^2 \alpha}{\sin \alpha} + \frac{\frac{\partial k_2}{\partial v}}{3\sqrt{G}(k_1 - k_2)} \frac{\sin^2 \alpha}{\cos \alpha} \right] \frac{\partial}{\partial \alpha}.$$

Consider the *Darboux plane field* defined by $\{\mathcal{D}_1, \mathcal{D}_2\}$.

First, consider the new pair of vector fields $\tilde{\mathcal{D}}_1 = \mathcal{D}_1 + \mathcal{D}_2$ and $\tilde{\mathcal{D}}_2 = \mathcal{D}_1 - \mathcal{D}_2$ and obtain:

$$\begin{aligned}\tilde{\mathcal{D}}_1 &= \frac{2 \cos \alpha}{\sqrt{E}} \frac{\partial}{\partial u} + \left[\frac{2 \partial k_1 / \partial u}{3\sqrt{E}(k_1 - k_2)} \frac{\cos^2 \alpha}{\sin \alpha} \right] \frac{\partial}{\partial \alpha} \\ \tilde{\mathcal{D}}_2 &= \frac{2 \sin \alpha}{\sqrt{G}} \frac{\partial}{\partial v} + \left[\frac{2 \partial k_2 / \partial v}{3\sqrt{G}(k_1 - k_2)} \frac{\sin^2 \alpha}{\cos \alpha} \right] \frac{\partial}{\partial \alpha}.\end{aligned}$$

Our Darboux plane \mathcal{D} is generated by:

$$\begin{aligned}\bar{\mathcal{D}}_1 &= \frac{2}{\sqrt{E}} \frac{\partial}{\partial u} + \left[\frac{2\partial k_1/\partial u}{3\sqrt{E}(k_1 - k_2)} \frac{\cos \alpha}{\sin \alpha} \right] \frac{\partial}{\partial \alpha} \\ \bar{\mathcal{D}}_2 &= \frac{2}{\sqrt{G}} \frac{\partial}{\partial v} + \left[\frac{2\partial k_2/\partial v}{3\sqrt{G}(k_1 - k_2)} \frac{\sin \alpha}{\cos \alpha} \right] \frac{\partial}{\partial \alpha}.\end{aligned}$$

Consider the unitary vector fields X_i , the conformal vector fields ξ_i and the principal conformal curvatures θ_i .

Observing that $\frac{\partial k_1/\partial u}{\sqrt{E}} = (X_1)k_1$ and $\frac{\partial k_2/\partial v}{\sqrt{G}} = (X_2)k_2$ we obtain a new base defined by:

$$\begin{aligned}\mathcal{D}_1^c &= \xi_1 + \frac{1}{6}\theta_1 \frac{\cos \alpha}{\sin \alpha} \frac{\partial}{\partial \alpha} \\ \mathcal{D}_2^c &= \xi_2 + \frac{1}{6}\theta_2 \frac{\sin \alpha}{\cos \alpha} \frac{\partial}{\partial \alpha}.\end{aligned}$$

This ends the proof of the first part.

Let $\Omega = Adu + Bdv + Cd\alpha$ be a one-form defining \mathcal{D} .

Solving the equations $\Omega(\mathcal{D}_1^c) = 0$ and $\Omega(\mathcal{D}_2^c) = 0$, making use of the definition of ω_1 and ω_2 , we find A , B and C providing our result. \square

Proposition 19. *We have that:*

$$\begin{aligned}\Omega \wedge d\Omega &= \sin \alpha \cos \alpha [-\theta_1\theta_2 + 3\xi_2(\theta_1) - 3\xi_1(\theta_2) \\ &\quad + 3 \cos 2\alpha (\xi_2(\theta_1) + \xi_1(\theta_2))] \omega_1 \wedge \omega_2 \wedge d\alpha.\end{aligned}\tag{11}$$

Proof. Let $d\theta_1 = \xi_1(\theta_1)\omega_1 + \xi_2(\theta_1)\omega_2$ and $d\theta_2 = \xi_1(\theta_2)\omega_1 + \xi_2(\theta_2)\omega_2$.

From the equation (7) it follows that:

$$\begin{aligned}d\theta_1 \wedge \omega_1 + \theta_1 d\omega_1 &= \left(\frac{\theta_1\theta_2}{2} + \xi_2(\theta_1) \right) \omega_1 \wedge \omega_2 \\ d\theta_2 \wedge \omega_2 + \theta_2 d\omega_2 &= \left(\frac{\theta_1\theta_2}{2} + \xi_1(\theta_2) \right) \omega_1 \wedge \omega_2\end{aligned}$$

Therefore,

$$\begin{aligned}d\Omega &= \left(\frac{\theta_1\theta_2}{2} - \xi_2(\theta_1) \cos^2 \alpha + \xi_1(\theta_2) \sin^2 \alpha \right) \omega_1 \wedge \omega_2 \\ &\quad + 2 \sin \alpha \cos \alpha (\theta_2 d\alpha \wedge \omega_2 - \theta_1 d\alpha \wedge \omega_1)\end{aligned}$$

A straightforward calculations leads to the result claimed. \square

Theorem 20. *The Darboux plane field \mathcal{D} is integrable if and only if*

$$(\xi_1)\theta_2 = -\frac{1}{6}\theta_1\theta_2, \quad (\xi_2)\theta_1 = \frac{1}{6}\theta_1\theta_2.\tag{12}$$

Here ξ_i are the conformal vector fields and θ_i are the principal conformal curvatures.

Proof. The theorem is a direct consequence of Proposition 19. In fact, by Frobenius theorem, the Darboux plane field \mathcal{D} is integrable if and only if $\Omega \wedge d\Omega = 0$. The equation

$$[-\theta_1\theta_2 + 3\xi_2(\theta_1) - 3\xi_1(\theta_2) + 3 \cos 2\alpha (\xi_2(\theta_1) + \xi_1(\theta_2))] = 0,$$

is equivalent to $-\theta_1\theta_2 + 3\xi_2(\theta_1) - 3\xi_1(\theta_2) = 0$ and $(\xi_2(\theta_1) + \xi_1(\theta_2)) = 0$. Direct calculations leads to the result stated in equation (12). \square

Next we will establish a partial relation between the integrability of the Darboux plane field with the property of being isothermic. In the case of canal surfaces a complete relation is established. Also an invariant measure associated to the principal foliations will be analyzed.

The class of isothermic surfaces was considered by Lamé, G. Darboux [Da4], P. Calapso [Ca] among others. For more recent works see for example [H-J] and references therein.

Ro: melhor retirar este comentario abaixo, percentiei, fev/2011 R: More or nothing

Definition 21. *A surface M is called isothermic if there is a locally conformal parameterization of the surface by principal curvature lines.*

Proposition 22. *Consider a surface M such that the Darboux plane field \mathcal{D} is integrable. Then M is isothermic.*

Proof. Let ξ_1 and ξ_2 be the principal conformal vector fields. As $|\xi_1| = |\xi_2| \neq 0$ a surface has a locally conformal parameterization by curvature lines if and only if there exists a function $h(u, v)$ such that $[h\xi_1, h\xi_2] = 0$.

Since $[\xi_1, \xi_2] = -\frac{1}{2}\theta_2\xi_1 - \frac{1}{2}\theta_1\xi_2$, direct calculation shows that

$$[h\xi_1, h\xi_2] = -h[\frac{1}{2}h\theta_2 + \xi_2(h)]\xi_1 + h[-\frac{1}{2}h\theta_1 + \xi_1(h)]\xi_2.$$

So the surface is isothermic when if there exists h such that

$$\xi_1(h) = \frac{1}{2}h\theta_1 \text{ and } \xi_2(h) = -\frac{1}{2}h\theta_2. \quad (13)$$

Developing the calculations and using the condition of integrability of plane field \mathcal{D} expressed by $\xi_1(\theta_2) + \xi_2(\theta_1) = 0$ we obtain $\xi_2(\xi_1(h)) = \xi_1(\xi_2(h))$. So the compatibility equation is satisfied and there is a solution h of 13 exists and the surface is isothermic. \square

Theorem 23. *Let M be a canal surface. Then M is isothermic if and only if the Darboux plane field \mathcal{D} is integrable.*

Proof. In a canal surface one of the conformal principal curvatures, say θ_2 vanishes identically. Therefore if M is isothermic, then $\xi_2(\theta_1) = 0$. The conditions of integrability of \mathcal{D} are given by: $-\theta_1\theta_2 + 3\xi_2(\theta_1) - 3\xi_1(\theta_2) = 0$ and $\xi_2(\theta_1) + \xi_1(\theta_2) = 0$. Therefore, when $\theta_2 = 0$ these two conditions are equivalent to $\xi_2(\theta_1) = 0$ and the result follows. The converse is given by Proposition 22. \square

To fix a notation, consider a local principal parameterization (u, v) of a surface M and let two leaves $\mathbf{F}(p_1)$ and $\mathbf{F}(p_2)$ of the principal foliation \mathcal{P}_2 passing through the points p_1 and p_2 . Let also $\mathbf{F}(p_1, p_2)$ be the leaf of \mathcal{P}_1 passing through these two points. Suppose that the orientation of the leaves is compatible with that of the surface.

Denote by $\pi_{1,2} : \mathbf{F}(p_1) \rightarrow \mathbf{F}(p_2)$ the transition map defined by the principal foliation \mathcal{P}_1 and by ds_2 the arc length of the leaves of \mathcal{P}_2 .

A similar construction for the other transition maps defined by \mathcal{P}_2 .

Proposition 24. *Let M be an isothermic surface, free of umbilical points. Then the principal foliation \mathcal{P}_1 preserves a transversal measure, i.e, there exists a positive function $H(p)$ such that $H(p)ds_2$ is preserved by all transition maps $\pi_{1,2}$ as defined above. In the special case when the mean curvature \mathcal{H} **R: check** of the surface is constant, the transversal measure is given by $\sqrt{(k_1 - k_2)/2} ds_2 = \sqrt{\mu} ds_2$.*

Proof. Under the hypothesis M is a torus and so is conformally equivalent to a flat torus $T^2 = \mathbb{C}/\{a_1\mathbb{Z} \oplus a_2\mathbb{Z}\}$. A fundamental domain is given by $\{z \in \mathbb{C} : z = aa_1 + ba_2, 0 \leq a, b \leq 1\}$. Let $\varphi : M \rightarrow T^2$ be the conformal equivalence. Then φ can be lifted to a map $\bar{\varphi}$ of the universal covering \mathbb{C} of both surfaces. The lifted map is an automorphism of \mathbb{C} and so is, modulo translation, of the form $z \mapsto \lambda z$ for some $\lambda \in \mathbb{C}$.

Also, as M is isothermic, the principal foliations can be lifted to \mathbb{C} and they are defined by two orthogonal commuting vector fields. This means that the two principal foliations can be defined by two exact one forms and so can be integrated. In geometric language this means that a transversal measure is preserved by all transition maps.

When \mathcal{H} is constant the transversal measure was computed in [G-S2] and it is exactly the conformal length of the leaves. This follows since the 1-form $\omega = dk_2/(k_2 - k_1) = dk_2/(2(\mathcal{H} - k_2))$ is exact when \mathcal{H} is constant. \square

Remark: It was established in [G-S1] that the derivative of the return map of a principal cycle γ of \mathcal{P}_1 is given by

$$\pi'(0) = \exp\left[-\int_{\gamma} \frac{dk_2}{k_2 - k_1}\right] = \exp\left[-\int_{\gamma} \frac{dk_1}{k_2 - k_1}\right].$$

In terms of conformal invariants (curvature θ_1 and length ds_c) it follows that

$$\pi'(0) = \exp\left[\int_{\gamma} \theta_1 ds_c\right], \quad ds_c = \mu ds.$$

The osculating sphere \mathcal{O}_C to a curve C at a point m is the unique sphere containing the osculating circle to the curve at the point which has a contact with the curve of order larger than the contact of the osculating circle with the curve. This shows that the osculating sphere is a conformally defined object.

5 Foliations making a constant angle with respect to Principal Foliations

In this section, we study cyclides and quadrics endowed with the foliations making a constant angle with the principal foliations.

Consider a surface M with principal foliations \mathcal{P}_1 and \mathcal{P}_2 and umbilic set \mathcal{U} . The triple $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{U})$ will be referred to as the *principal configuration* of the surface.

Ro: complement of definition, fev/2011

Definition 25. For each angle $\alpha \in (-\pi/2, \pi/2)$ we can consider the foliations \mathcal{F}_α^+ and \mathcal{F}_α^- such that the leaves of this foliation are the curves making a constant angle $\pm\alpha$ with the leaves of the principal foliation \mathcal{P}_1 . We will write $\mathcal{F}_\alpha = \{\mathcal{F}_\alpha^+, \mathcal{F}_\alpha^-, \mathcal{U}\}$ to denote this configuration.

Remark: The definition of \mathcal{F}_α involves only an angle with \mathcal{P}_1 , therefore it is a conformal one. Notice also that the direction of \mathcal{F}_α^\pm are the direction tangent at m to the intersection of the sphere σ_α (see Proposition 4) with M .

The normal curvature of a leaf of \mathcal{F}_α is precisely $k_n(\alpha) = k_1 \cos^2 \alpha + k_2 \sin^2 \alpha$.

5.1 Foliation \mathcal{F}_α on Dupin cyclides

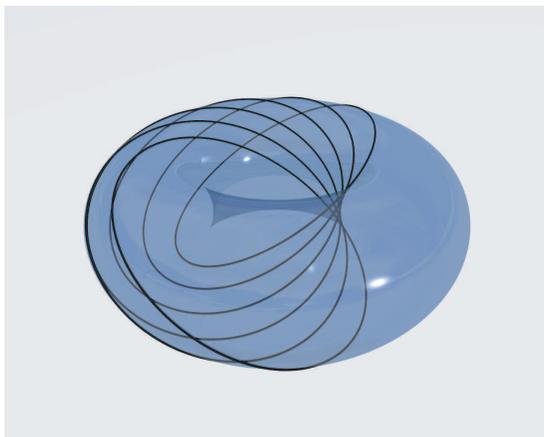


Figure 11: Foliation of a torus of revolution by Villarceau circles

Dupin cyclides are very special: they are surfaces which are in two different ways envelopes of one-parameter families of spheres (see [Da3]). This implies that the corresponding curves are circles or hyperbolas in Λ^4 , intersection of Λ^4 with an affine plane (see [La-Wa]).

There are three types of Dupin cyclides. One can chose a nice representant of each class:

- A) the boundary of a tubular neighborhood of a geodesic of \mathbb{S}^3 ,
- B) a cylinder of revolution in \mathbb{R}^3 ,
- C) a cone of revolution in \mathbb{R}^3 .

Then, in cases A) and B) the foliations \mathcal{F}_α^+ and \mathcal{F}_α^- are totally geodesic. In case A), four of them are consist of circles: the two foliations by characteristic circles \mathcal{F}_0 and $\mathcal{F}_{\pi/2}$, and the two others by Villarceau circles. Recall that the angle of the Villarceau circles with principal foliations depends on the radius of the tubular neighborhood.

In the case C), on can develop the cone on a plane. This procedure provides a local isometry out of the origin . In the plane, one can see that a foliation by curves making a

constant angle with rays is a foliation by logarithmic spirals. The picture on the cone is obtained by rolling the planar foliation back on the cone.

5.2 Foliations \mathcal{F}_α on quadrics

In this subsection we describe the global behavior of the foliations \mathcal{F}_α on the ellipsoid. On the hyperboloid and on the paraboloid, the triple orthogonal system and the foliations \mathcal{F}_α are simpler.

The quadric surfaces have many remarkable geometric properties. Some were already considered by D'Alembert [d'A], who was the first to observe that the ellipsoid has two families of circular sections. After that Monge and Hachette [Mo3] showed that all generic quadric surfaces have two families of circular sections. Recall that Monge [Mo1, Mo3] also described the global behavior of principal lines on the ellipsoid. Probably, this is the origin of the theory of singular foliations on surfaces.

\mathcal{R} : why here? We just use lines of curvature. What about the lines, particular circles on the hyperboloid?

Jacobi ([Ja]), in the nineteenth century, studied the geodesic flow on the ellipsoid, see [Kl] and [Ga-S3]. For a sample of caustics on quadrics see [Be].

Proposition 26. *Consider an ellipsoid $\mathbb{E}_{a,b,c} = \{(x, y, z) : h(x, y, z) = \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1\}$ with $a > b > c > 0$. Then $\mathbb{E}_{a,b,c}$ have four umbilical points located in the plane of symmetry orthogonal to the middle axis. They are singularities of the foliations by curvature lines which have index $1/2$ and one separatrix (type D_1).*

For all $0 < \alpha < \frac{\pi}{2}$, the singularities of \mathcal{F}_α are the four umbilical points and the corresponding configuration is locally topologically equivalent to the principal configuration \mathcal{P} of the ellipsoid near an umbilic point.

\mathcal{R} o: an alternative proof: ainda nao consegui finalizar. Estou tentando fazer uma prova mais geometrica

Proof. Consider the parameterization of the ellipsoid in a neighborhood of the umbilical point $p_0 = (x_0, 0, z_0) = (\sqrt{\frac{a(a-b)}{a-c}}, 0, \sqrt{\frac{c(b-c)}{a-c}})$.

$$\beta(u, v) = p_0 + uE_1 + vE_2 + \frac{1}{2}[\sqrt{ac}(u^2 + v^2) + \frac{\sqrt{c(b-c)(a-b)}}{b^3}(u^3 + uv^2) + h.o.t]E_3$$

Here $\{E_1, E_2, E_3\}$, $E_2 = (0, 1, 0)$, is a positive orthonormal base; the ellipsoid is oriented by $E_3 = -\nabla h(p_0)/|\nabla h(p_0)|$, where ∇h is the gradient of h .

In a neighborhood of the umbilical point $(0, 0)$, the differential equation of the configuration \mathcal{F}_α in the chart (u, v) is given by $A(u, v)dv^2 + B(u, v)dudv + C(u, v)du^2 = 0$, where

$$A(u, v) = -u - \cos 2\alpha \sqrt{u^2 + v^2 + O(r^3)} + O(r^2),$$

$$B(u, v) = 2v + O(r^2),$$

$$C(u, v) = u - \cos 2\alpha \sqrt{u^2 + v^2 + O(r^3)} + O(r^2),$$

and $r = \sqrt{u^2 + v^2}$.

The above implicit differential equation has two real separatrices with limit directions given by $\pm 2\alpha$. The behavior of the integral curves near 0 is the same of an umbilical point

of type D_1 . In fact, consider the blowing-up $u = r \cos \theta$, $v = r \sin \theta$. The differential equation of \mathcal{F}_α in the new variables is given by:

$$\begin{aligned} & (\cos 2\alpha - \cos \theta + rR_1(r, \theta))dr^2 + r(2 \sin \theta + rR_2(r, \theta))drd\theta \\ & + r^2(\cos 2\alpha + \cos \theta + rR_3(r, \theta))d\theta^2 = 0. \end{aligned}$$

The two singular points are given by $r = 0, \theta = \pm 2\alpha$. Direct analysis shows that both singular points are hyperbolic saddles of the vector fields adapted to the implicit equation near these singularities. The blowing-down of the saddle separatrices are the separatrices of \mathcal{F}_α^+ and \mathcal{F}_α^- , see Fig. 12.

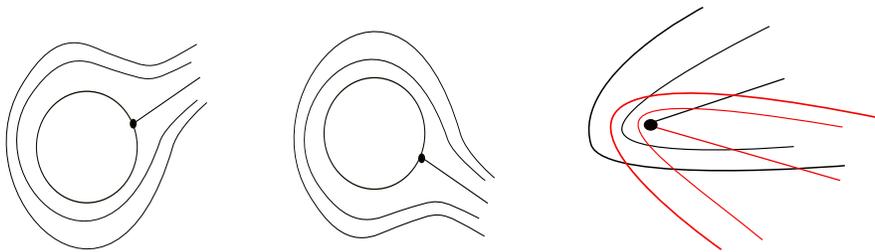


Figure 12: Blowing-up the foliations \mathcal{F}_α^+ and \mathcal{F}_α^- .

Therefore, the pair of foliations \mathcal{F}_α^+ and \mathcal{F}_α^- near an umbilical point of our ellipsoid is locally topologically equivalent to the configuration of principal lines near a Darbouxian umbilical point of type D_1 of index $1/2$ having exactly one hyperbolic sector. \square

Remark: Near an isolated umbilical point the generic behavior of principal lines, on real analytic surfaces, was established by Darboux, [Da2]. See also [Gu], [G-S1], [Ga-S3] and references therein.

Ro: nesta proposicao esta correto o termo linear nos coeficientes a, b e c. Usei a convencao do Spivak. fiz nova redacao da proposicao e da demonstracao.

Proposition 27. Consider an ellipsoid $\mathbb{E}_{a,b,c}$ and principal coordinates (u, v) with $b \leq u \leq a$ and $c \leq v \leq b$. On the ellipse $\Sigma_{xz} \subset \mathbb{E}_{a,b,c}$, containing the four umbilical points, p_i , ($i = 1, \dots, 4$) counterclockwise oriented, denote by $s_1(\alpha) = 2 \int_c^b \sin \alpha \left[\frac{\sqrt{v}}{-H(v)} \right] dv$ (resp. $s_2(\alpha) = 2 \int_b^a \cos \alpha \left[\frac{\sqrt{u}}{H(u)} \right] du$, $H(t) = (t - a)(t - b)(t - c)$) evaluated between the adjacent umbilical points p_1 and p_4 (resp. p_1 and p_2). Define $\rho(\alpha) = \frac{s_2(\alpha)}{s_1(\alpha)}$.

Then if $\rho \in \mathbb{R} \setminus \mathbb{Q}$ (resp. $\rho \in \mathbb{Q}$) all the leaves of \mathcal{F}_α^+ and \mathcal{F}_α^- are dense (resp. all, with the exception of the umbilic separatrices, are closed). See Fig. 13.

Proof. The ellipsoid $\mathbb{E}_{a,b,c}$ belongs to the Dupin triple orthogonal system of surfaces defined by the one parameter family of quadrics, $\frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} + \frac{z^2}{c-\lambda} = 1$ with $a > b > c > 0$, see also [Sp] and [St]. The following parameterization $\beta(u, v) = (x(u, v), y(u, v), z(u, v))$ of $\mathbb{E}_{a,b,c}$, where

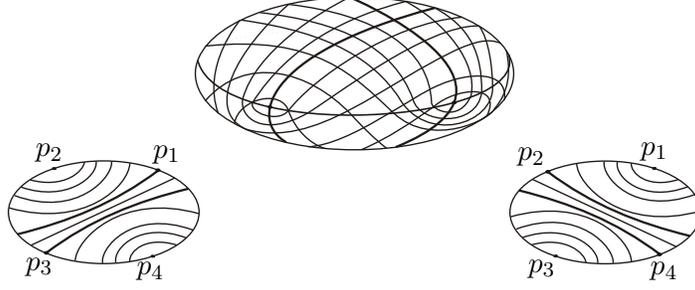


Figure 13: Foliations \mathcal{F}_α^\pm of the ellipsoid $\mathbb{E}_{a,b,c}$

$$\beta(u, v) = \left(\pm \sqrt{\frac{a(u-a)(v-a)}{(b-a)(c-a)}}, \pm \sqrt{\frac{b(u-b)(v-b)}{(b-a)(b-c)}}, \pm \sqrt{\frac{c(u-c)(v-c)}{(c-a)(c-b)}} \right), \quad (14)$$

defines the principal coordinates (u, v) on $\mathbb{E}_{a,b,c}$, with $u \in (b, a)$ and $v \in (c, b)$.

The first fundamental form in the chart (u, v) of $\mathbb{E}_{a,b,c}$ is given by:

$$I = ds^2 = Edu^2 + Gdv^2 = \frac{(v-u)u}{4H(u)} du^2 + \frac{(u-v)v}{4H(v)} dv^2 \quad (15)$$

The second fundamental form with respect to the normal $N = -(\beta_u \wedge \beta_v) / |\beta_u \wedge \beta_v|$ is given by

$$II = edu^2 + gdv^2 = \frac{(v-u)}{4H(u)} \sqrt{\frac{abc}{uv}} du^2 + \frac{(u-v)}{4H(v)} \sqrt{\frac{abc}{uv}} dv^2 \quad (16)$$

where $H(t) = (t-a)(t-b)(t-c)$. **℞o: troquei o H anterior por h e mantive a notacao aqui**

Therefore the principal curvatures are given by:

$$k_1 = \frac{e}{E} = \frac{1}{u} \sqrt{\frac{abc}{uv}}, \quad k_2 = \frac{g}{G} = \frac{1}{v} \sqrt{\frac{abc}{uv}}.$$

The four umbilical points are given by:

$$(\pm x_0, 0, \pm z_0) = \left(\pm \sqrt{\frac{a(a-b)}{a-c}}, 0, \pm \sqrt{\frac{a(c-b)}{c-a}} \right)$$

The differential equation of the configuration \mathcal{F}_α in the principal chart (u, v) is given by

$$\begin{aligned} H(u)v \cos^2 \alpha dv^2 + H(v)u \sin^2 \alpha du^2 &= 0 \Leftrightarrow \\ \frac{v}{H(v)} \cos^2 \alpha dv^2 + \sin^2 \alpha \frac{u}{H(u)} du^2 &= 0. \end{aligned}$$

Define $d\sigma_1 = \sin \alpha \sqrt{-\frac{u}{H(u)}} du$ and $d\sigma_2 = \cos \alpha \sqrt{\frac{v}{H(v)}} dv$.

Therefore the differential equation of the configuration \mathcal{F}_α is equivalent to $d\sigma_2^2 - d\sigma_1^2 = 0$ in the rectangle $[0, s_1(\alpha)] \times [0, s_2(\alpha)]$.

On the ellipse $\Sigma = \{(x, y, z) | \{\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1, y = 0\}$ define a non euclidean distance between the umbilical points $p_1 = (x_0, 0, z_0)$ and $p_4 = (x_0, 0, -z_0)$ by $s_1(\alpha) = 2 \int_c^b \sin \alpha [\frac{\sqrt{v}}{-H(v)}] dv$ and that between the umbilical points $p_1 = (x_0, 0, z_0)$ and $p_2 = (-x_0, 0, z_0)$ is given by $s_2(\alpha) = 2 \int_b^a [\cos \alpha \frac{\sqrt{u}}{H(u)}] du$.

The ellipse Σ is the union of four umbilical points and four principal umbilic separatrices for the principal foliations. So $\Sigma \setminus \{p_1, p_2, p_3, p_4\}$ is a transversal section of the configuration \mathcal{F}_α .

Therefore near the umbilical point p_1 the foliation, say \mathcal{F}_α^+ , with umbilic separatrix contained in the region $\{y > 0\}$ define a the return map $\sigma_+ : \Sigma \rightarrow \Sigma$ which is an isometry, reverting the orientation, with $\sigma_+(p_1) = p_1$. This follows because in the principal chart (u, v) this return map is defined by $\sigma_+ : \{u = b\} \rightarrow \{v = b\}$ which satisfies the differential equation $\frac{ds_2}{ds_1} = -1$. By analytic continuation it results that σ_+ is a isometry reverting orientation with two fixed points $\{p_1, p_3\}$. The geometric reflection σ_- , defined in the region $y < 0$ have the two umbilic $\{p_2, p_4\}$ as fixed points. So the Poincaré return map $\pi_1 : \Sigma \rightarrow \Sigma$ (composition of two isometries σ_+ and σ_-) is a rotation with rotation number given by $s_2(\alpha)/s_1(\alpha)$.

Analogously for \mathcal{F}_α^- with the Poincaré return map given by $\pi_2 = \tau_+ \circ \tau_-$ where τ_+ and τ_- are two isometries having respectively $\{p_2, p_4\}$ and $\{p_1, p_3\}$ as fixed points. \square

Remark: The special case $\alpha = \pi/4$ was studied in [Ga-S1]. A more general framework of implicit differential equations, unifying various families of geometric curves was studied in [Ga-S2]. See also [Ga-S3].

Proposition 28. *In any surface, free of umbilical points, the leaves of \mathcal{F}_α^+ and \mathcal{F}_α^- are Darboux curves if and only if the surface is conformal to a Dupin cyclide.*

Proof. From the differential equation of Darboux curves, see equation (9) in Section 3.2, it follows that:

$$3(k_1 - k_2) \sin \alpha \cos \alpha \frac{d\alpha}{ds} = \frac{1}{\sqrt{E}} \frac{\partial k_1}{\partial u} \cos^3 \alpha + \frac{1}{\sqrt{G}} \frac{\partial k_2}{\partial v} \sin^3 \alpha.$$

So all Darboux curves are leaves of \mathcal{F}_α^\pm if and only if $\frac{\partial k_1}{\partial u}(u, v) = \frac{\partial k_2}{\partial v}(u, v) = 0$. By Proposition 37 in Section 8 this is exactly the condition that characterizes the Dupin cyclides. \square

6 Darboux curves in cyclides: a geometric viewpoint

First notice that, on a regular Dupin cyclide, we know already some Darboux curves/ the Villarceau circles. The spheres containing a Villarceau circle form a pencil, therefore correspond to points of a geodesic circle in Λ^4 . This curve is therefore also a geodesic in $V(M)$. The spheres tangent to M at points of the Villarceau circle form an arc on this circle; this proves that the Villarceau circle is a Darboux curve.

We can describe $V(M)$ when the surface M is a regular Dupin cyclide (see Figure 14). It is the wedge of the two circles formed by the osculating spheres of the regular cyclide.

Consider the principal configuration $\mathcal{P} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{U})$ of a surface M .

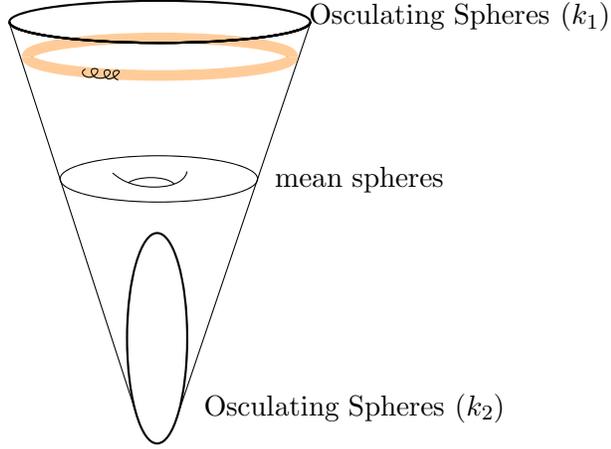


Figure 14: Darboux curves in $V(M)$ when M is a regular Dupin cyclide

Let us recall (see Definition 25) that for each angle $\alpha \in (-\pi/2, \pi/2)$ we can consider the foliations \mathcal{F}_α^+ and \mathcal{F}_α^- such the leaves of this foliation are the curves making a constant angle $\pm\alpha$ with the leaves of the principal foliation \mathcal{P}_1 .

Let us, for later use, define the surfaces $M_\alpha \subset V(M)$ as the set of spheres tangent to M at a point m having curvature $k_\alpha = k_n(\alpha) = k_1 \cos^2 \alpha + k_2 \sin^2 \alpha$. Each surface M_α is foliated by the lifts of the curves of \mathcal{F}_α ; we call this new foliation $\tilde{\mathcal{F}}_\alpha$. These later foliation form a foliation of $V(M)$ that we call $\tilde{\mathcal{F}}$.

Proposition 29. *The Darboux curves in Dupin cyclides are the leaves of the foliations \mathcal{F}_α^+ and \mathcal{F}_α^- .*

Proof: - *Case a) Cylinders.* this is the easiest case to visualize. Cylinders are models for cyclides having exactly one singular point.

The definition of a Darboux curve requires that the osculating sphere to the helix \mathcal{H} drawn on the cylinder of revolution is tangent to it. This comes from the fact that the helix is invariant by the rotation R of angle π and axis equal to the principal curvature vector of the helix which is a vector orthogonal to the cylinder and going along its axis of revolution. As the osculating sphere should also be invariant by this rotation its diameter is contained in the line normal to the cylinder which is the axis of the rotation. Therefore the sphere is tangent to the cylinder.

- *Case b) Regular cyclides in \mathbb{S}^3* Each regular Dupin cyclide M is, for a suitable metric of constant curvature 1, the tubular neighborhood of a geodesic \mathcal{G}_1 of \mathbb{S}^3 (see [La-Wa]). Seeing \mathbb{S}^3 as the unit sphere of the Euclidian space \mathbb{E}^4 of dimension 4, the geodesic is the intersection of a 2-plane P_1 with \mathbb{S}^3 . Let P_2 be the plane orthogonal to P_1 in \mathbb{E}^4 . Then it is also a tubular neighborhood of the geodesic $\mathcal{G}_2 = P_2 \cap \mathbb{S}^3$. We can define (for this metric) the symmetries with respect to the two spheres containing respectively m and \mathcal{G}_1 and m and \mathcal{G}_2 . Let \mathcal{R} be the composition of these symmetries. Then \mathcal{R} preserves the cyclide but also the “helices”, that is leaves of \mathcal{F}_α on the cyclide. It should therefore also preserve the osculating sphere to the helix, which, as it is not normal to the cyclide, has to be tangent to it. The “helix” is therefore a Darboux curve. There are enough “helices”

to be sure we got all the Darboux curves.

-*Case c) Cones.* The last family of Dupin cyclides is formed of conformal images of cones of revolution (see [La-Wa]). One can deal with cones of revolution as we did with regular cyclides, using a sphere orthogonal to the axis and m (it belongs to the pencil whose limit points are the singular points) and a sphere containing the axis and m . \square We will, using the general dynamical properties of Darboux curves, prove (see Proposition 28) that a surface is a Dupin cyclide if and only if its Darboux curves are the leaves of the foliation $\tilde{\mathcal{F}}$ of $V(M)$.

7 Darboux curves near a Ridge Point

In this section the semi-local dynamical behavior of Darboux curves it will be developed. More precisely, we will analyze the asymptotic behavior of Darboux curves when the curve becomes tangent to a principal direction and will consider the qualitative behavior of Darboux curves near a regular curve of ridge points which is transversal to the correspondent principal foliation. We will establish two different patterns, zigzag and beak-to-beak, see Fig. 17. The ridge points are associated to inflections of the principal curvature lines, to the singularities of the focal set of the surface and also to the singularities of the boundary of $V(M)$ (see 2.1). We refer the reader to [Po] for an introduction to ridges and to [Gu] for application to the construction of eye-lenses.

Consider a surface M and a principal chart (u, v) such that the horizontal foliation \mathcal{P}_1 is that associated to the principal curvature k_1 .

Definition 30. A non umbilical point $p_0 = (u_0, v_0)$ is called a ridge point of the principal foliation \mathcal{P}_1 if $\frac{\partial k_1}{\partial u}(p_0) = 0$, equivalently, if $\theta_1(p_0) = 0$. In the way we define ridges for \mathcal{P}_2 .

Definition 31. A ridge point p_0 relative to \mathcal{P}_1 is called zigzag, when $\sigma_1(p_0) = \frac{\partial^2 k_1}{\partial u^2}(p_0)/(k_1(p_0) - k_2(p_0)) < 0$, equivalently when $\xi_1(\theta_1(p_0)) < 0$.

It is called beak-to-beak, when $\sigma_1(p_0) > 0$, equivalently when $\xi_1(\theta_1(p_0)) > 0$.

We define zigzag and beak-to-beak points for \mathcal{P}_2 in the same way.

The type of contact of the osculating sphere with curvature k_i with the surface at a ridge point p_0 is determined by the analytic conditions $\theta_i(p_0) = 0$ and $\xi_i(\theta_i(p_0)) \neq 0$, $i=1,2$. In the zigzag case, we have a center contact and in the beak-to-beak case, the contact is of saddle type, see Fig. 6. These contacts can be represented canonically by the contacts between the plane $z = 0$ and the surfaces $z = x^4 \pm y^2 = 0$ or equivalently $z = x^2 \pm y^4 = 0$.

When the surface is parameterized by a graph, a practical way to characterize the type of a ridge point is given by the following proposition.

Proposition 32. Consider a surface of class C^r , $r \geq 4$ parameterized by the graph $(u, v, h(u, v))$ where

$$h(u, v) = \frac{k_1}{2}u^2 + \frac{k_2}{2}v^2 + \frac{a}{6}u^3 + \frac{d}{2}u^2v + \frac{b}{2}uv^2 + \frac{c}{6}v^3 + \frac{A}{24}u^4 + \frac{B}{6}u^3v + \frac{C}{4}u^2v^2 + \frac{D}{6}uv^3 + \frac{E}{24}v^4 + h.o.t \quad (17)$$

Then $(0, 0)$ is a ridge point for \mathcal{P}_1 when $a = 0$. The osculating sphere at the ridge point writes $\sigma_1 = \frac{A-3k_1^3}{k_1-k_2} + \frac{2d^2}{(k_1-k_2)^2}$.

In the same way $(0, 0)$ is a ridge point for \mathcal{P}_2 when $c = 0$. Then $\sigma_2(0) = \frac{E-3k_2^3}{k_2-k_1} + \frac{2b^2}{(k_2-k_1)^2}$.

Proof. Straightforward calculations shows that the principal curvatures in a neighborhood of $(0, 0)$ are given by:

$$\begin{aligned} k_1(u, v) &= k_1 + au + dv + \frac{1}{2}(A - 3k_1^3 + \frac{2d^2}{k_1 - k_2})u^2 + (B - 2\frac{bd}{k_2 - k_1})uv \\ &\quad + \frac{1}{2}(C - k_1k_2^2 - \frac{2b^2}{k_2 - k_1})v^2 + h.o.t. \\ k_2(u, v) &= k_2 + bu + cv + \frac{1}{2}(C - k_1^2k_2 + \frac{2d^2}{k_2 - k_1})u^2 + (D + 2\frac{bd}{k_2 - k_1})uv \\ &\quad + \frac{1}{2}(E - 3k_2^3 + \frac{2b^2}{k_2 - k_1})v^2 + h.o.t. \end{aligned} \tag{18}$$

The result follows. \square

Proposition 33. *Let p_0 be a point of a ridge of M corresponding to the principal foliation \mathcal{P}_1 such that $\sigma_1(p_0) \neq 0$. Then the ridge R containing p_0 is locally a regular curve transverse to \mathcal{P}_1 ; the boundary of $V(M)$ corresponding to $\alpha = 0$ has a cuspidal edge along $\pi^{-1}(R)$, see Fig. 15. The same is true for ridges associated to the principal foliation \mathcal{P}_2 .*

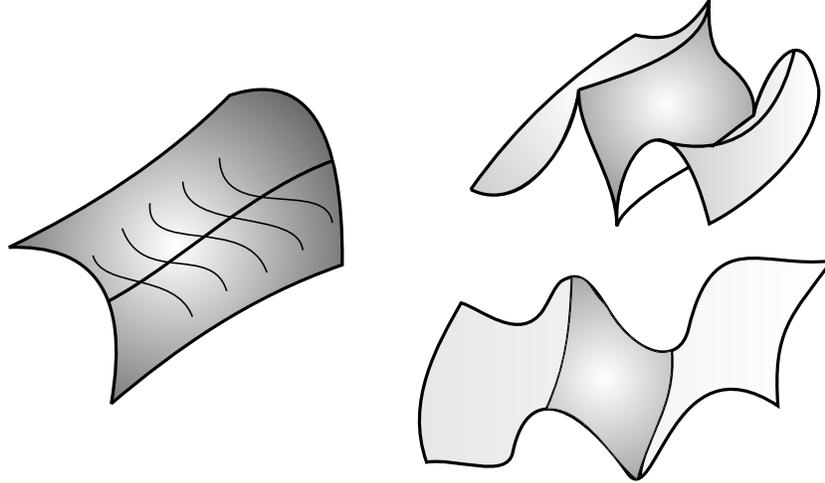


Figure 15: Ridges and Singularities of the boundary of $V(M)$

Proof. In the parameterization given in Proposition 32, at $p_0 = (0, 0)$, the principal direction corresponding to \mathcal{P}_1 is $e_1 = (1, 0)$ and the ridge is given (see Equation (18)) by

$$u(v) = \left(-\frac{d(k_1 - k_2)}{(A - 3k_1^3)(k_1 - k_2) + 2d^2}v + O(v^2)\right). \tag{19}$$

Next, consider a principal chart (u, v) . The set of spheres $V(M) \subset \Lambda^4$ is parameterized by $\varphi(u, v, \alpha) = k_n(\alpha)m(u, v) + N(u, v)$ with $k_n(\alpha) = k_1(u, v) \cos^2 \alpha + k_2(u, v) \sin^2 \alpha$, $\mathcal{L}(m, m) = 0$ and $N_u = -k_1 m_u$, $N_v = -k_2 m_v$, see equation (1).

The boundary of $V(M)$ is parameterized by $\alpha = 0$ and $\alpha = \pi/2$ and so $\varphi_1(u, v) = \varphi(u, v, 0) = k_1(u, v)m(u, v) + N(u, v)$ and $\varphi_2(u, v) = \varphi(u, v, \frac{\pi}{2}) = k_2(u, v)m(u, v) + N(u, v)$. The map φ_1 and φ_2 have rank 1 at the corresponding ridge points. Therefore we recognize cuspidal edges on the boundary of $V(M)$. \square

Proposition 34. *Let p_0 be a non-umbilic not in a ridge. A Darboux curve tangent to a principal line at p_0 makes a cusp which can be parameterized by*

$r_1(t) = (\frac{3}{2} \frac{\partial k_1}{\partial u} (k_1 - k_2) t^2 + \dots, \frac{\partial k_1}{\partial u} (k_1 - k_2) t^3 + \dots)$ or $r_2(t) = (\frac{\partial k_2}{\partial v} (k_2 - k_1) t^3 + \dots, \frac{3}{2} \frac{\partial k_2}{\partial v} (k_2 - k_1) t^2 + \dots)$. The behavior of all the Darboux curves passing through p_0 is as shown in Fig. 16.

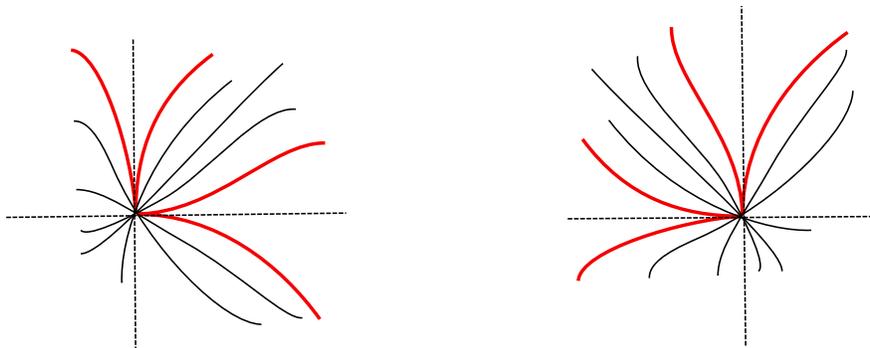


Figure 16: Darboux curves through a non ridge point

Proof. Consider the vector field X defined by the differential equations:

$$\begin{aligned} u' &= \frac{1}{\sqrt{E}} \cos \alpha [3(k_1 - k_2) \sin \alpha \cos \alpha], \\ v' &= \frac{1}{\sqrt{G}} \sin \alpha [3(k_1 - k_2) \sin \alpha \cos \alpha], \\ \alpha' &= \frac{1}{\sqrt{E}} \frac{\partial k_1}{\partial u} \cos^3 \alpha + \frac{1}{\sqrt{G}} \frac{\partial k_2}{\partial v} \sin^3 \alpha. \end{aligned}$$

The projections of the integral curves of X in the coordinates (u, v) are precisely the Darboux curves.

For any initial condition $(0, 0, \alpha_0)$, with $\alpha_0 \neq n\pi/2, n \in \mathbb{Z}$, the integral curves of X are transverse to the axis α and therefore have regular projections. For $\alpha_0 = n\pi/2$ and $\sigma_2 \neq 0$, the integral curves of X are tangent to the axis α and the projections are of cuspidal type. For $\alpha = n\pi$ direct calculations provide

$$(u(t), v(t)) = \left(\frac{3}{2} \frac{\partial k_1}{\partial u} (k_1 - k_2) t^2 + \dots, (-1)^n \frac{\partial k_1}{\partial u} (k_1 - k_2) t^3 + \dots \right).$$

Now observe that the projection of the integral curves of X passing through $(0, 0, 0)$ and $(0, 0, \pi)$ form a cusp tangent to a line of curvature. **Ro: aceitei a modificacao proposta** \square

Theorem 35. *Let R be an arc of a ridge transverse to the corresponding principal foliation (i.e., suppose that $\sigma_i(p) \neq 0$ for every $p \in R$). Then there exist exactly two types of behavior for the Darboux curves near the ridge, zigzag and beak-to-beak.*

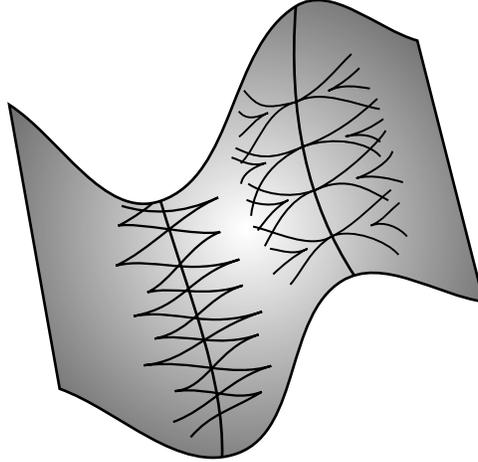


Figure 17: Darboux curves near regular curve of ridges: zigzag and beak-to-beak.

Proof. Consider the vector field X as in the proof of Proposition 34.

We will consider a ridge corresponding to \mathcal{P}_1 . For the other principal foliation the analysis is similar.

The ridge is defined by the equation $\frac{\partial k_1}{\partial u}(u, v) = 0$.

In the parametrization $(u, v, h(u, v))$ introduced in equation 17 the singularities of $X(u, v, \alpha)$ are defined by $(u(v), v, 0)$, see equation 19.

Ro: veja correcao acima R: What are U(v) and V(u)?

To simplify the notation suppose a singular point $(0, 0, 0)$ **R: (0,0,0)in \mathbb{R}^3 where the surface is of in (u, v, α) coordinates?** of X , the ridge transversal to the principal foliation \mathcal{P}_1 .

It follows that:

$$DX(0) = \begin{pmatrix} 0 & 0 & 3(k_1 - k_2) \\ 0 & 0 & 0 \\ \frac{\partial^2 k_1}{\partial u^2} & 0 & 0 \end{pmatrix}$$

The eigenvalues of $DX(0)$ are:

$$\lambda_1 = 0, \lambda_2 = \frac{1}{\sqrt{3}} \sqrt{\frac{\partial^2 k_1}{\partial u^2} / (k_1 - k_2)}, \lambda_3 = -\frac{1}{\sqrt{3}} \sqrt{\frac{\partial^2 k_1}{\partial u^2} / (k_1 - k_2)}.$$

By invariant manifold theory, when $\lambda_2 \lambda_3 = -\frac{1}{3} \frac{\partial^2 k_1}{\partial u^2} / (k_1 - k_2) = -\frac{1}{3} \sigma_1(0) < 0$, the singular set of X (ridge set) is normally hyperbolic and there are stable and unstable

surfaces, normally hyperbolic along the singular set. This implies that there is a lamination (continuous fibration) along the ridge set and the fibers are the Darboux curves. Also the prolonged Darboux curves are of class C^1 along the ridge set.

So the Darboux curves are as shown in Fig. 17, right. That is, there are Darboux curve crossing the ridge, tangent to the principal lines, and the prolonged Darboux curves are C^1 along the ridge set.

In the case when $\sigma_1(0) = \frac{\partial^2 k_1}{\partial u^2} / (k_1 - k_2) < 0$ the non zeros eigenvalues of $DX(0)$ are purely complex and so the singular set is not normally hyperbolic.

In this case we are in the hypothesis of Roussarie Theorem, [Ro, Theorem 20, page 59], so there is a local first integral in a neighborhood of the ridge set. The level sets of this first integral are cylinders and the integral curves (like helices) in each cylinder when projected in the surface M has a cuspidal point exactly when the helix cross the section $\alpha = 0$. This produces the zigzag as shown in Fig. 17, left.

There are no Darboux curves tangent to the principal direction $e_1 = \partial/\partial u$ along the ridge set in this case. \square

8 Darboux curves on general cylinders, cones and surfaces of revolution

The Darboux curves on general cones were already studied by Santaló ([Sa2]). In a similar way, one can study Darboux curves on cylinders and surfaces of revolution. This is not a coincidence. The three types of surfaces are canals corresponding to a curve $\gamma \subset \mathbb{L}^4$ which is also contained in a 3-dimensional subspace of \mathbb{L}^5 . Depending on the subspace, this intersection is either a copy of \mathbb{L}^2 , a unit sphere \mathbb{S}^3 or a 2-dimensional cylinder (see [Da1], [M-N], [Ba-La-Wa]). The latter condition defines conformal images of general cones, general cylinders and surfaces of revolution.

These surfaces can be obtained imposing conformally invariant local conditions.

Canal surface are characterized locally, [Da1], [H-J], [M-N], [Ba-La-Wa], by the following propositions.

Proposition 36. *A surface M is (a piece of) a canal if and only if one of its conformal principal curvatures, say θ_2 , is equal to zero.*

Since Dupin cyclides are the only surfaces which canal in two different ways, they can be characterized by the condition $\theta_1 = \theta_2 = 0$.

Proposition 37. *A surface such that $\theta_2 = 0$ and θ_1 is constant along characteristic circles can be obtained as the image by a Möbius map of a cone, a cylinder or a surface of revolution.*

The surfaces characterized in Proposition 37 are called *special canal surfaces* in [Ba-La-Wa].

Proposition 38. *Let M be a special canal surface and (u, v) be a principal chart such that $\theta_1(u, v) = \theta_1(u)$ and $\theta_2(u, v) = 0$.*

Let $A(u) = \exp[\int \frac{k'_1}{k_1 - k_2} du]$ and $\alpha \in (0, \pi)$ be an angle. Then the function $\mathcal{J}(u, \alpha) = A(u) \cos^3 \alpha$ is a first integral of the Darboux curves. Moreover in the region $A_c = \pi(M_c) =$

$\{(u, v) : u \in M_c\}$, $M_c = \mathcal{J}^{-1}(c)$, the Darboux curves are defined by the implicit differential equation

$$c^{2/3}Gdv^2 - E(A^{2/3} - c^{2/3})du^2 = 0.$$

Proof. The differential equation (9) reduces to

$$u' = \frac{\cos \alpha}{\sqrt{E}}, \quad v' = \frac{\sin \alpha}{\sqrt{G}}, \quad \alpha' = \frac{1}{3\sqrt{E}} \frac{k_1'}{k_1 - k_2} \frac{\cos^2 \alpha}{\sin \alpha}.$$

Therefore, $\frac{d\alpha}{du} = \frac{1}{3} \frac{k_1'}{k_1 - k_2} \frac{\cos \alpha}{\sin \alpha}$ which is an equation where the variables are separable. Direct integration leads to the first integral \mathcal{J} as stated.

To obtain the implicit differential equation solve the equation $\mathcal{J}(u, v) = c$ in function of $\cos \alpha$ and observe that $\frac{dv}{du} = \frac{\sqrt{E} \sin \alpha}{\sqrt{G} \cos \alpha}$. \square

Proposition 39. *Let M be a special canal surface. Then the Darboux plane field \mathcal{D} is integrable.*

Proof. Direct from the characterization of integrability of \mathcal{D} established in Theorem 20. \square

8.1 Cylinders

The case of cylinders is the simplest. Let $C = \{c(u)\}$ be a plane curve of curvature $k(u)$. The cylinder of axis generated by a vector \vec{z} orthogonal to the plane can be parameterized by $\phi(u, v) = c(u) + v\vec{z}$. The function $\mathcal{I}(u, \alpha) = k(u) \cos^3 \alpha$ is a first integral for Darboux curves.

8.2 Cones

Proposition 40. *The Darboux curves on a cone, free of umbilical points (that is without flat points), can be integrated by quadratures. The function*

$$\mathcal{I}(u, \alpha) = k_g(u) \cos^3 \alpha$$

is a first integral of the differential equation of Darboux curves. Here k_g is the geodesic curvature of the intersection of the cone with the unitary sphere.

Moreover, if $k_g'/k_g < 0$ the ridge is zigzag. If $k_g'/k_g > 0$ the ridge is beak-to-beak. See Fig. 18.

Proof. The cone can be parameterized by $X(u, v) = v\gamma(u)$ where $|\gamma| = 1$ and $|\gamma'| = 1$ is a spherical curve.

Since $\gamma'' = -\gamma + k_g\gamma \wedge \gamma'$ and $N(u, v) = \gamma \wedge \gamma'$, $k_1(u, v) = k_g(u)$ and $k_2(u, v) = 0$,

Therefore, Darboux curves are given by $(\sin \alpha / \cos \alpha)d\alpha = \frac{1}{3}(k_g'/k_g)du$ and $\mathcal{I}(u, \alpha) = k_g(u) \cos^3 \alpha$ is a first integral for them. The behavior of Darboux curves near ridges follows from Theorem 35. \square

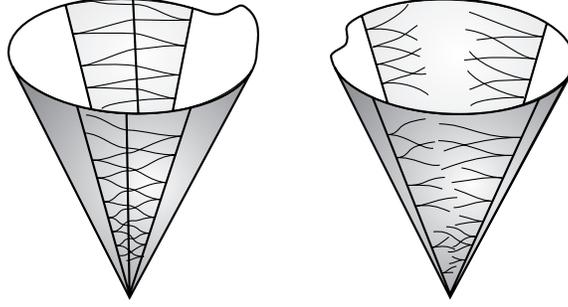


Figure 18: Darboux curves near regular ridges on a general cone

8.3 Surfaces of revolution

Traditionally, a surface of revolution is defined from a profile. Here we view it as a canal surface obtained from a one parameter family of spheres of radii $r(u)$ and centers at $(0, 0, u)$ on the vertical axis.

The envelope of this family is a canal surface and can be parameterized by:

$$H(u, v) = r(u) \cos \beta(u) (\cos v, \sin v, 0) + (0, 0, u - r(u) \sin \beta(u)),$$

where $\cos \beta(u) = \sqrt{1 - r'(u)^2}$, $\sin \beta = r'$, $|r'(u)| < 1$, $\beta \in (-\pi/2, \pi/2)$.

The unit normal to the surface is

$$N = (-\cos \beta(u) \cos v, -\cos \beta(u) \sin v, \sin \beta(u)).$$

The coefficients of the first and second fundamental forms of H are given by

$$E(u, v) = \frac{(1 - r'^2 - rr'')^2}{1 - r'^2}, \quad F(u, v) = 0, \quad G(u, v) = r^2(1 - r'^2)$$

$$e(u, v) = -\frac{r''(1 - r'^2 - rr'')}{1 - r'^2}, \quad f(u, v) = 0, \quad g(u, v) = r(1 - r'^2).$$

The principal curvatures are given by

$$k_1(u, v) = -\frac{r''}{1 - r'^2 - rr''} \quad k_2(u, v) = \frac{1}{r}.$$

It will be assumed that the surface is free of umbilical points, for example that $k_2 > k_1$. Ridges are defined by the equation $R(u) = 0$, where

$$R(u) = \frac{\partial k_1}{\partial u} = r'''(1 - r'^2) + 3r'r''^2 = 0.$$

Proposition 41. *Darboux curves on surfaces of revolution, free of umbilical points, can be obtained from integration by quadratures. The function*

$$\mathcal{I}(u, \alpha) = r(u) \cos \beta(u) (k_1 - k_2) \cos^3 \alpha = h(u) (k_1 - k_2) \cos^3 \alpha,$$

R: we wrote $r(u)$, $\beta(u)$, OK? is a first integral of the differential equation of Darboux curves, where h is the distance of the point of the surface to the axis of revolution.

Moreover, if $R'(u) < 0$ the ridge is zigzag. If $R'(u) > 0$ the ridge is beak-to-beak, see Fig. 20.

The functions h , r and β are related by $h = r \cos \beta$.

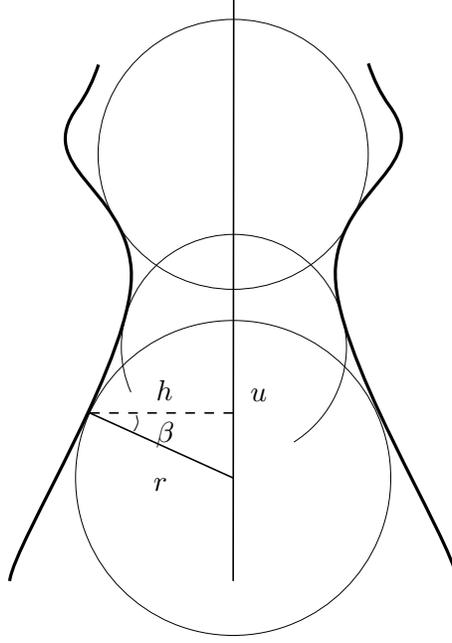


Figure 19: A surface of revolution as a canal

Proof. In the principal chart (u, v) the differential equation of Darboux curves is given by

$$u' = \frac{1}{\sqrt{E}} \cos \alpha, \quad v' = \frac{1}{\sqrt{G}} \sin \alpha, \quad \alpha' = \frac{1}{3\sqrt{E}} \left[\frac{k_1'}{k_1 - k_2} \right] \frac{\sin^2 \alpha}{\cos \alpha}$$

Therefore, it follows that $3 \frac{\sin \alpha}{\cos \alpha} d\alpha = \frac{k_1'}{k_1 - k_2} du$.

Now observe that

$$\begin{aligned} \int \frac{k_1'}{k_1 - k_2} du &= \int \frac{k_1' - k_2'}{k_1 - k_2} du + \int \frac{k_2'}{k_1 - k_2} du = \ln(k_2 - k_1) + \int \frac{k_2'}{k_1 - k_2} du \\ \int \frac{k_2'}{k_1 - k_2} du &= \int \left[\frac{r'}{r} + \frac{r'r''}{1 - r'^2} \right] du = \ln r (1 - r'^2)^{1/2} \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{I}(u, v, \alpha) &= r(1 - r'^2)^{1/2} (k_1 - k_2) \cos^3 \alpha \\ &= r \cos \beta (k_1 - k_2) \cos^3 \alpha = h(u) (k_1 - k_2) \cos^3 \alpha \end{aligned}$$

is the first integral. Again, the behavior of Darboux curves near ridges follows from Theorem 35. \square

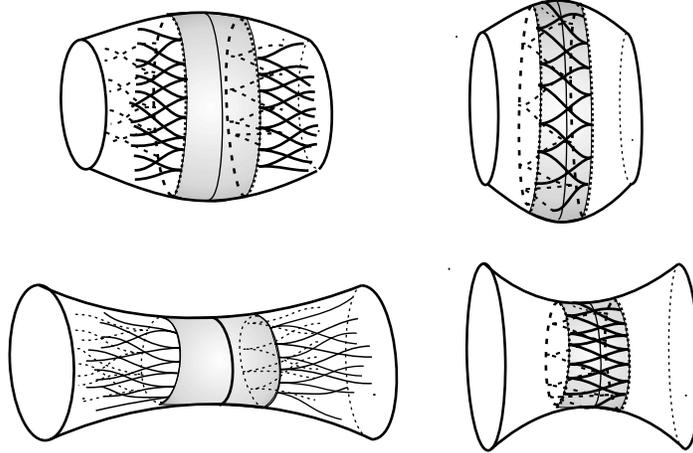


Figure 20: Darboux curves near regular ridges on surfaces of revolution

First notice that the osculating spheres of curvature k_1 of the surface of revolution intersect a plane containing the axis of revolution in the osculating circles of the profile. They form a surface $\mathcal{O}_1 \subset \Lambda^4$ (see 7). The osculating spheres of radius k_2 intersect the same plane in the circle of center $(0, u)$ and radius $r(u)$. They form a curve $\gamma \subset \Lambda^2 \subset \Lambda^4$.

Ridges on a surface of revolution are characteristic circles. They correspond to vertices of the profile. They also correspond to the singularities of the surface $\mathcal{O}_1 \subset \Lambda^4$ (see Figure 7).

Here the osculating spheres to the surface along a meridian form a curve δ in a totally geodesic $\Lambda^3 \subset \Lambda^4$. We can also see this curve as corresponding to the osculating circles of the profile. The vertices of the profile correspond to singular points of δ . At these singular points the tangent direction is light-like. The one-parameter family of rotations g_t leaving the surface of revolution invariant extend to a one-parameter family of isometries of the Lorentz space \mathbb{R}_1^4 such that $\mathcal{O}_1 = \cup g_t(\delta)$.

R: I put % in front of the remark about geodesics. Why was it here?

9 Darboux curves on quadrics

A geometrical aspect of the theory of quadrics is the following classical result. Given three straight lines L_1, L_2 and L_3 in \mathbb{R}^3 in general position the set of lines intersecting these lines defines a one parameter family of lines which generate a unique twice ruled surface. This was first established by G. Monge [Mo3]. See also Spivak [Sp]. The basic idea is observe that this configuration is of projective nature and also can be extended to the projective space \mathbb{P}^3 . Taking coordinates such that the lines L_i are parallel to the axis and writing the equation of a general line L as the intersection of two planes $y = ax + b, z = cx + d$ it follows, from algebraic manipulations, that the set of lines which intercept the three given lines is a quadric, the hyperboloid of one sheet or the hyperbolic paraboloid.

Ro: In higher dimension see [Fl].

We already noticed, studying the Darboux curves on Dupin cyclides, that a circle (or a line) on a surface is a Darboux curve. In fact, we can chose a sphere tangent to the

surface in the pencil of spheres containing the circle or the line. The corresponding curve in Λ^4 is a geodesic, therefore it is a geodesic in $V(M)$, proving that the initial curve is a Darboux curve.

As a quadric Q with pairwise distinct principal axes contains two families of circles, the corresponding 3-manifold $V(Q)$ contains a surface filled with two families of geodesics arcs of the pencils of spheres containing the previous circles. The previous analysis of twice rules surfaces in \mathbb{P}^3 shows that the latter surface is a piece of the intersection of Λ^4 with a quadratic cone of \mathbb{R}_1^4 .

Next it will be developed the qualitative analysis of Darboux curves on the quadrics $\mathbb{Q}_{a,b,c}$.

The quadrics $\mathbb{Q}_{a,b,c}$ belongs to the triple orthogonal system of surfaces defined by the one parameter family of quadrics, $\frac{x^2}{a-\lambda} + \frac{y^2}{b-\lambda} + \frac{z^2}{c-\lambda} = 1$ with $a > b > c$, see also [Sp] and [St].

The Darboux curves in the ellipsoid were considered by A. Pell in [Pe] and J. Hardy in [Ha]. Here we complete and hopefully simplify these works, describing the global behavior of Darboux curves in a triple orthogonal system of quadrics.

Consider the principal chart (u, v) and the parameterization of $\mathbb{Q}_{a,b,c}$ given by equation (14) in Section 5.

For the ellipsoid, with $a > b > c > 0$, $u \in (b, a)$, $v \in (c, b)$ or $u \in (c, b)$, $v \in (b, a)$.

For the hyperboloid of one sheet, with $a > b > 0 > c$, $u \in (b, a)$, $v < c$ or $u < c$, $v \in (b, a)$.

For the hyperboloid of two sheets, with $a > 0 > b > c$, $u \in (c, b)$, $v < c$ or $u < c$, $v \in (c, b)$.

The first fundamental form of $\mathbb{Q}_{a,b,c}$ is given by equation (15) and the second is given by equation (16).

Therefore the principal curvatures are given by:

$$k_1 = \frac{e}{E} = \frac{1}{u} \sqrt{\frac{abc}{uv}}, \quad k_2 = \frac{g}{G} = \frac{1}{v} \sqrt{\frac{abc}{uv}}.$$

Also consider the functions

$$r_1 = \frac{\partial k_1}{\partial u} / (3(k_1 - k_2)) = \frac{1}{2} \frac{v}{u(u-v)}, \quad r_2 = \frac{\partial k_2}{\partial v} / (3(k_1 - k_2)) = \frac{1}{2} \frac{u}{v(u-v)}.$$

The four umbilical points are $(\pm x_0, 0, \pm z_0) = (\pm \sqrt{\frac{a(a-b)}{a-c}}, 0, \pm \sqrt{\frac{c(b-c)}{a-c}})$.

Proposition 42. *The differential equation of Darboux curves on a quadric $\mathbb{Q}_{a,b,c}$ is given by:*

$$\begin{aligned} u' &= \frac{1}{\sqrt{E}} \cos^2 \alpha \sin \alpha, & v' &= \frac{1}{\sqrt{G}} \cos \alpha \sin^2 \alpha \\ \alpha' &= \frac{r_1}{\sqrt{E}} \cos^3 \alpha + \frac{r_2}{\sqrt{G}} \sin^3 \alpha \end{aligned} \tag{20}$$

Proof. In a principal chart the differential equation of Darboux curves is given by equation (9). Define $\tan \alpha = \frac{\sqrt{G}v'}{\sqrt{E}u'}$ so that $Eu'^2 + Gv'^2 = 1$. To obtain a regular extension of the differential equation (9) to $\alpha = m\pi/2$ and consider it as a vector field in the variables (u, v, α) multiply the resultant equation by the factor $\cos \alpha \sin \alpha$ and the result follows. \square

Proposition 43. *The function*

$$I(u, v, u', v') = I(u, v, \alpha) = \frac{\cos^2 \alpha}{u} + \frac{\sin^2 \alpha}{v}, \quad \tan \alpha = \frac{v'}{u'} \sqrt{\frac{G(u, v)}{E(u, v)}} \quad (21)$$

is a first integral of equation (20).

Proof. We have that $I(u, v, \alpha) = \frac{\cos^2 \alpha}{u} + \frac{\sin^2 \alpha}{v}$.

Let $c(s) = (u(s), v(s), \alpha(s))$ be a solution of the differential equation (20). It will shown that $\frac{d}{ds}(I(u(s), v(s), \alpha(s))) = 0$.

We have that

$$E_s = \frac{1}{4} \frac{uw'}{H(u)} - \frac{1}{4} \frac{u'}{H(u)^2} [(u-2v)H(u) + (uv-u^2)H'(u)]$$

$$G_s = \frac{1}{4} \frac{vu'}{H(v)} - \frac{1}{4} \frac{v'}{H(v)^2} [(u-2v)H(v) + (v^2-uv)H'(v)]$$

Straightforward calculation leads to $\frac{d}{ds}(I(u(s), v(s), u'(s), v'(s))) = 0$.

□

Proposition 44. *The Darboux curves on a quadric $\mathbb{Q}_{a,b,c}$ are the real integral curves of the implicit differential equation:*

$$(v-\lambda)H(u)v'^2 - (u-\lambda)H(v)u'^2 = 0, \quad H(x) = (x-a)(x-b)(x-c).$$

The normal curvature in a direction \mathcal{D}

\mathcal{R} : conflict of notations with the Darboux plane field

defined by $I(u, v, dv/du) = 1/\lambda$ is given by $k_n(p, \mathcal{D}) = \frac{1}{\lambda} \sqrt{\frac{abc}{uv}}$.

This differential equation is equivalent to

$$k_n(u, v, [du : dv]) = \frac{e(u, v)du^2 + g(u, v)dv^2}{E(u, v)du^2 + G(u, v)dv^2} = \frac{1}{\lambda} \sqrt{\frac{abc}{uv}} = \frac{1}{\lambda} (abc)^{1/4} \mathcal{K}^{1/4}.$$

Proof. From the equation $I = 1/\lambda$ it follows that

$$\left(\frac{dv}{du}\right)^2 = \frac{(u-\lambda)vE(u, v)}{(v-\lambda)uG(u, v)} = \frac{(u-\lambda)H(v)}{(v-\lambda)H(u)}.$$

Here dv/du is the direction \mathcal{D} defined by $I(u, v, dv/du) = 1/\lambda$.

As $k_n = \frac{e+g(dv/du)^2}{E+G(dv/du)^2}$ it follows from the equation above and from equations (15) and

(16) that $k_n(p, \mathcal{D}) = \frac{1}{\lambda} \sqrt{\frac{abc}{uv}} = \frac{1}{\lambda} (abc)^{1/4} \mathcal{K}^{1/4}$, $\mathcal{K} = k_1 k_2$.

For the reciprocal part consider the implicit differential equation

$$\frac{e(u, v)du^2 + g(u, v)dv^2}{E(u, v)du^2 + G(u, v)dv^2} = \frac{1}{\lambda} \sqrt{\frac{abc}{uv}}.$$

Using equations (15) and (16) it follows that this equation is equivalent to the following.

$$\frac{\lambda-u}{H(u)} du^2 - \frac{\lambda-v}{H(v)} dv^2 = 0 \quad \Leftrightarrow \quad (u-\lambda)H(v)du^2 - (v-\lambda)H(u)dv^2 = 0.$$

The restriction in the values of λ is in order to consider only real solutions of the implicit differential equation obtained. □

Proposition 45. *The ridge set of the quadric $\mathbb{Q}_{a,b,c}$ is the intersection of the quadric with the coordinates planes. Moreover:*

a) *For the ellipsoid with $0 < c < b < a$ it follows that:*

- i) *The ellipse $E_{xy} = \{z = 0\} \cap \mathbb{E}_{a,b,c}$, respectively $E_{yz} = \{x = 0\} \cap \mathbb{E}_{a,b,c}$, is a ridge corresponding to k_2 , respectively to k_1 , and is zigzag.*
- ii) *The ellipse $E_{xz} = \{y = 0\} \cap \mathbb{E}_{a,b,c}$ containing the four umbilical points $(\pm x_0, 0, \pm z_0)$ is the union of ridges of k_1 and k_2 . For $|x| > x_0$ the ridge correspond to k_1 . The ellipse E_{xz} is beak-to-beak in both cases.*

b) *For the hyperboloid of one sheet with $c < 0 < b < a$ it follows that:*

- i) *The hyperbole H_{yz} is a ridge corresponding to k_1 and is beak-to-beak.*
- ii) *The hyperbole H_{xz} is a ridge corresponding to k_1 and is zigzag.*
- iii) *The ellipse E_{xy} is a ridge corresponding k_2 and it is zigzag.*

c) *For the hyperboloid of two sheets with $c < b < 0 < a$ it follows that:*

- i) *The hyperbole H_{xz} is a ridge corresponding to k_1 and is zigzag.*
- ii) *The hyperbole H_{xy} containing the four umbilical points $(\pm x_0, \pm y_0, 0)$ is the union of ridges of k_1 and k_2 . For $|x| > x_0$ the ridge correspond to k_1 and all segments of hyperbolas are beak-to-beak.*

Proof. a) Ellipsoid: By symmetry is clear that the points of intersection of the coordinates planes with the ellipsoid are ridges points. The principal curvatures have no critical points along the corresponding principal curvature line in the complement of these three ellipses. In a principal chart (u, v) we have that $\frac{dk_1}{du} = -\frac{3}{2u}k_1 \neq 0$ and $\frac{dk_2}{dv} = -\frac{3}{2v}k_2 \neq 0$.

It will be sufficient to check the condition of zigzag or beak-to-beak only in a point of a connected component of the ridge set.

Consider the point $p_0 = (-\sqrt{a}, 0, 0)$. The ellipsoid is parametrized by:

$$x(y, z) = -\sqrt{a} + \sqrt{a}[\frac{y^2}{2b} + \frac{z^2}{2c} + \frac{y^4}{8b^2} + \frac{y^2z^2}{4bc} + \frac{z^2}{8c^2} + h.o.t.]$$

Therefore $k_1(p_0) = \sqrt{a}/b$, $k_2(p_0) = \sqrt{a}/c$, $A = 3\sqrt{a}/b^2$ (A is the coefficient of y^4) and $((A - 3k_1^3)(k_2 - k_1) = -3a(a - b)(b - c)/(b^4c) < 0$. Therefore the ellipse E_{xz} is beak-to-beak.

Also, let $E = 3\sqrt{a}/c^2$ (coefficient of z^4). So, $(E - 3k_2^3)(k_1 - k_2) = 3a(a - c)(b - c)/(bc^4) > 0$. Therefore, by Theorem 35 the ellipse E_{xy} is zigzag.

Now consider the point $q_0 = (0, -\sqrt{b}, 0)$. The ellipsoid is parametrized by:

$$y(x, z) = -\sqrt{b} + \sqrt{b}[\frac{x^2}{2a} + \frac{z^2}{2c} + \frac{x^4}{8a^2} + \frac{x^2z^2}{4ac} + \frac{z^2}{8c^2} + h.o.t.]$$

Now, with A coefficient of x^4 and E coefficient of z^4 it follows that $(A - 3k_1^3)(k_2 - k_1) = 3b(a - b)(a - c)/(a^4c) > 0$ and $(E - 3k_2^3)(k_1 - k_2) = 3b(b - c)(a - c)/(ac^4) > 0$.

So both ellipses E_{yz} and E_{xy} are both zigzag, one for the corresponding principal curvature.

b) Hyperboloid of one sheet: the ridges are given by the intersection of the hyperboloid with the coordinates planes.

Consider the point $q_0 = (0, -\sqrt{b}, 0)$. The hyperboloid is parametrized by:

$$y(x, z) = -\sqrt{b} + \sqrt{b} \left[\frac{x^2}{2a} + \frac{z^2}{2c} + \frac{x^4}{8a^2} + \frac{x^2 z^2}{4ac} + \frac{z^2}{8c^2} + h.o.t. \right]$$

Therefore $k_1(q_0) = \sqrt{b}/a > 0$ $k_2(q_0) = \sqrt{b}/c < 0$, $A = 3\sqrt{b}/a^2$ (A is the coefficient of x^4) and $E = 3\sqrt{b}/c^2$ coefficient of z^4 it follows that $(A - 3k_1^3)(k_1 - k_2) = -3b(a - b)(a - c)/(a^4c) > 0$ and $(E - 3k_2^3)(k_2 - k_1) = -3b(b - c)(a - c)/(ac^4) < 0$.

The hyperbole H_{yz} is beak-to-beak and the ellipse E_{xy} is zigzag.

Next consider the point $p_0 = (-\sqrt{a}, 0, 0)$. The hyperboloid is parametrized by:

$$x(y, z) = -\sqrt{a} + \sqrt{a} \left[\frac{y^2}{2b} + \frac{z^2}{2c} + \frac{y^4}{8b^2} + \frac{y^2 z^2}{4bc} + \frac{z^2}{8c^2} + h.o.t. \right]$$

Therefore $k_1(p_0) = \sqrt{a}/b > 0$ $k_2(p_0) = \sqrt{a}/c < 0$, $A = 3\sqrt{a}/b^2$ (A is the coefficient of y^4) $E = 3\sqrt{a}/c^2$ (coefficient of z^4) it follows that $((A - 3k_1^3)(k_1 - k_2) = 3a(a - b)(b - c)/(b^4c) < 0$ and $(E - 3k_2^3)(k_2 - k_1) = -3a(a - c)(b - c)/(bc^4) < 0$. Therefore the hyperbole H_{xz} and the ellipse E_{xy} are both zigzag.

c) Hyperboloid of two sheets: the ridges are the intersection of the hyperboloid with the coordinates planes.

Consider the point $p_0 = (\sqrt{a}, 0, 0)$. One leaf of the hyperboloid is parametrized by:

$$x(y, z) = \sqrt{a} - \sqrt{a} \left[\frac{y^2}{2b} + \frac{z^2}{2c} + \frac{y^4}{8b^2} + \frac{y^2 z^2}{4bc} + \frac{z^2}{8c^2} + h.o.t. \right]$$

Therefore $k_1(p_0) = -\sqrt{a}/b > 0$ $k_2(p_0) = -\sqrt{a}/c > 0$, $A = -3\sqrt{a}/b^2$ (A is the coefficient of y^4) and $((A - 3k_1^3)(k_1 - k_2) = 3a(a - b)(a - c)/(b^4c) < 0$. Therefore the hyperbole H_{xz} is zigzag.

Also, let $E = -3\sqrt{a}/c^2$ (coefficient of z^4). So, $(E - 3k_2^3)(k_2 - k_1) = -3a(a - c)(b - c)/(bc^4) > 0$. Therefore the hyperbole H_{xy} for $|x| < x_0$ is beak-to-beak.

For $|x| > x_0$ on the hyperbole H_{xy} a similar analysis shows that it is beak-to-beak. \square

Proposition 46. Consider the ellipsoid $\mathbb{E}_{a,b,c}$ with $a > b > c > 0$.

- i) For $c < \lambda < b$ the Darboux curves are and contained in cylindrical region $c < v < \lambda$ and the behavior is as in Fig. 21, upper left.
- ii) For $\lambda = b$ the Darboux curves are the circular sections of the ellipsoid. These circles are contained in planes parallels to the tangent plane to $\mathbb{E}_{a,b,c}$ at the umbilical points. These circles are tangent along the ellipse E_y and through each umbilical point pass only one Darboux curve. See Fig. 21, bottom right.
- iii) For $b < \lambda < a$ the Darboux curves are bounded contained in the two cylindrical region $\lambda \leq u \leq a$ and the behavior is as shown in the Fig. 21, upper right.

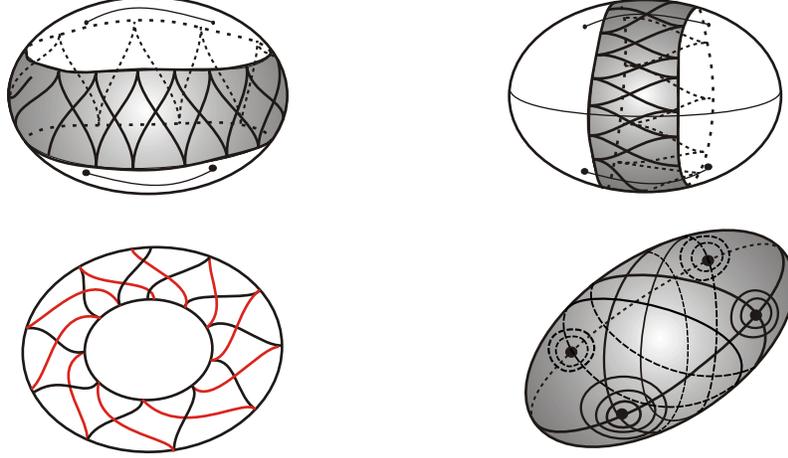


Figure 21: Darboux curves on the ellipsoid.

Proof. First case: $c < \lambda < b$.

The differential equation of Darboux curves is given by:

$$\frac{(v - \lambda)}{H(v)}v'^2 - \frac{(u - \lambda)}{H(u)}u'^2 = 0, \quad c \leq v \leq \lambda \quad \text{and} \quad b \leq u \leq a.$$

Define $d\sigma_1 = \sqrt{(u - \lambda)/H(u)}du$ and $d\sigma_2 = \sqrt{(v - \lambda)/H(v)}dv$.

Therefore the differential equation is equivalent to $d\sigma_1^2 - d\sigma_2^2 = 0$, with $(\sigma_1, \sigma_2) \in [0, L_1] \times [0, L_2]$ ($L_1 = \int_b^a d\sigma_1 < \infty$, $L_2 = \int_c^\lambda d\sigma_2 < \infty$).

In the ellipsoid this analysis implies the following.

The cylindrical region $C_\lambda = \alpha([b, a] \times [c, \lambda])$ is foliated by the integral curves of an implicit differential equation having cusp singularities in ∂C_λ . We observe that this region is free of umbilical point and is bounded by principal curvature lines, in coordinates defined by $v = c$ and $v = \lambda$.

The case $b < \lambda < a$ the analysis is similar. Now the differential equation of Darboux curves are defined in the region $[\lambda, a] \times [c, b]$ and we have a cylindrical region $C_\lambda = \alpha([\lambda, a] \times [c, b])$.

For $\lambda = b$ the differential equation can be simplified in the following.

$$(u - a)(u - c)dv^2 - (v - a)(v - c)du^2 = 0.$$

This equation is well defined in the rectangle $[c, a] \times [c, a]$ which contains the $[b, a] \times [c, b]$.

Define $d\sigma_1 = 1/\sqrt{(u - a)(u - c)}du$ and $d\sigma_2 = 1/\sqrt{(v - a)(v - c)}dv$. So the equation is equivalent $d\sigma_1^2 - d\sigma_2^2 = 0$ with $(\sigma_1, \sigma_2) \in [0, L] \times [0, L]$ ($L = \int_c^a d\sigma_1$). So in this rectangle all solutions are straight lines. The images of this family of curves on the ellipsoid are its circular sections. In fact we know that the ellipsoid has circular sections parallel to the tangent planes at umbilical points. As the circles are always Darboux curves it follows that the solutions of the differential equation is the family of circular sections. So we have two families of circles having tangency along the ellipse E_y . \square

Remark: In Fig. (21) a circle through an umbilic pass through the antipodal umbilical point only when $b^2 = (a^2 + c^2)/2$.

Proposition 47. Consider an ellipsoid $\mathbb{E}_{a,b,c}$ with three axes $a > b > c > 0$ and suppose $b < \lambda < a$. Let $L_1 := \int_b^a \sqrt{E(u,b)} du$ and $L_2 := \int_c^\lambda \sqrt{G(b,\lambda)} du$ and define $\rho = \frac{L_2}{L_1}$. Consider the Poincaré map $\pi : \Sigma \rightarrow \Sigma$ associated to the foliation of Darboux curves defined by the implicit differential equation $I = 1/\lambda$.

Then if $\rho \in \mathbb{R} \setminus \mathbb{Q}$ (resp. $\rho \in \mathbb{Q}$) all orbits are recurrent (resp. periodic) on the cylinder region $v \leq \lambda$. See Figure 21, bottom left.

Proof. The differential equation of Darboux curves is given by:

$$\frac{(v-\lambda)}{H(v)}v'^2 - \frac{(u-\lambda)}{H(u)}u'^2 = 0, \quad c \leq v < \lambda < u \leq a.$$

Define $d\sigma_1 = \sqrt{\frac{(u-\lambda)}{H(u)}} du$ and $d\sigma_2 = \sqrt{\frac{(v-\lambda)}{H(v)}} dv$. By integration, this leads to the chart (σ_1, σ_2) , in a rectangle $[0, L_1] \times [0, L_2]$ in which the differential equation of Darboux is given by

$$d\sigma_1^2 - d\sigma_2^2 = 0.$$

The proof ends with the analysis of the rotation number of the above equation. See similar analysis in Propositions 27 and 47. \square

Proposition 48. Consider a connect component of a hyperboloid of two sheets $\mathbb{H}_{a,b,c}$ with $a > 0 > b > c$.

- i) For $\lambda < c$ the Darboux curves are non bounded and contained in the non bounded region $v < \lambda$ and the behavior is as in the Fig. 22, left.
- ii) For $\lambda = c$ the Darboux curves are the circular sections of the hyperboloid. See Fig. 22, center.
- iii) For $c < \lambda < b$ the Darboux curves are non bounded and contained in the cylindrical region $\lambda \leq u \leq b$ and the local behavior is as shown in Fig. 22, right.

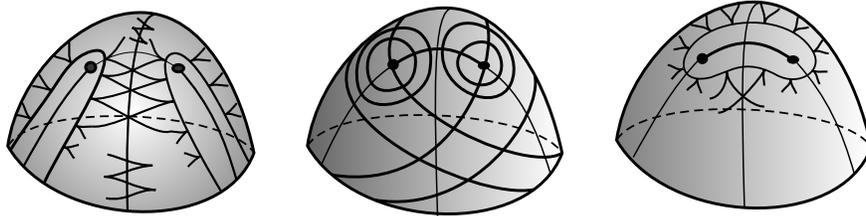


Figure 22: Darboux curves on a connected component of a hyperboloid of two sheets.

Proof. The analysis developed in the case of the ellipsoid also works here. See proof of Proposition 46. \square

Proposition 49. Consider an hyperboloid of one sheet $\mathbb{H}_{a,b,c}$ with $a > b > 0 > c$. Let $\lambda \in (-\infty, c) \cup (b, \infty)$.

- i) For $\lambda < c$ the Darboux curves are bounded and contained in the cylindrical region $\lambda \leq v \leq c$ and the behavior is as in Fig. 23, upper center.
- ii) For $b < \lambda < a$ the Darboux curves are unbounded and contained in the cylindrical region $b \leq u \leq \lambda$ (outside the hyperbola E_x) and the behavior is as in Fig. 23, upper right.
- iii) For $\lambda = a$ the Darboux curves are the circular sections of the hyperboloid. See Fig. 23, bottom left.
- iv) For $a < \lambda$ all Darboux curves are regular (helices) and go to ∞ in both directions. See Fig. 23, bottom right.
- v) The families of straight lines of the hyperboloid are not in the level sets of the first integral. See Fig. 23, upper left.

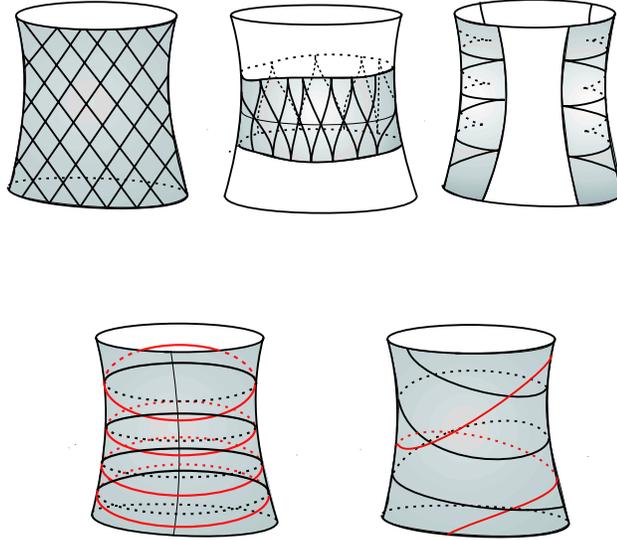


Figure 23: Darboux curves on a hyperboloid of one sheet.

Proof. Similar to the proof of Proposition 46. We observe that the hyperboloid of one sheet has two special families of Darboux curves, the straight lines and the circular sections. The family of circular sections are given by $\lambda = a$ and that of straight lines are given by $\lambda = \pm\infty$. \square

Remark: The circular sections of the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ of one sheet are given by the intersection of the surface with the planes defined by:

$$\begin{aligned}
 \pi_1(y, z, u) &= \frac{\sqrt{a^2 - b^2}}{b}y + \frac{\sqrt{a^2 + c^2}}{c}z - u = 0 \\
 \pi_2(y, z, v) &= -\frac{\sqrt{a^2 - b^2}}{b}y + \frac{\sqrt{a^2 + c^2}}{c}z - v = 0
 \end{aligned} \tag{22}$$

where $u, v \in \mathbb{R}$.
of the hyperboloid.

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References

- [d'A] J. d'Alembert, *Opuscles mathématiques ou Mémoires sur différens sujets de géométrie, de mécanique, d'optique, d'astronomie*. Tome VII, (1761), p. 163.
- [Ba-La-Wa] A. Bartoszek, R. Langevin and P.G. Walczak. *Special canal surfaces of \mathbb{S}^3* . Bull. Braz. Math. Soc. New Series **42**(2), (2011), pp. 301–320.
- [Be] M. Berger, *Seules les quadriques admettent des caustiques*. Bull. Soc. Math. France **123** (1995), pp. 107–116.
- [CSW] G. Cairns, R. Sharpe, L. Webb, *Conformal invariants for curves and surfaces in three dimensional space forms*, Rocky Mountain Jour. of Math. **24**, (1994), pp.933–959.
- [Ca] P. Calapso, *Sulla superficie a linee di curvatura isoterme*, Rend. Circ. Mat. Palermo, **17** (1903), pp. 275–286.
- [Co] E. Cosserat, *Sur les courbes tracés sur une surface et dont la sphère osculatrice est tangente en chaque point à la surface*, Note Comptes Rendus Acad. Scien. Paris, **121** (1895), pp. 43–46.
- [Da1] G. Darboux, *Des courbes tracées sur une surface, dont la sphère osculatrice est tangente en chaque point à la surface*, Note Comptes Rendus Acad. Scien. Paris, tome LXXIII (1872), pp. 732–736.
- [Da2] G. Darboux, *Sur la forme des lignes de courbure dans la voisinage d'un ombilic*. Leçons sur la Théorie des Surfaces, Vol. III, Gauthiers-Villars, (1887).
- [Da3] G. Darboux, *Leçons sur la théorie générale des surfaces*, Gauthier-Villars Paris (1887).
- [Da4] G. Darboux, *Sur les surfaces isothermiques*, Ann. Sci. École Norm. Sup. **16** (1899), pp. 491–508
- [En] A. Enneper, *Bemerkungen über die Differentialgleichung einer Art von Curven auf Flächen*, Göttinger Nachrichten (1891), pp. 577–583.
- [Fl] L.A. Florit, *Doubly ruled submanifolds in space forms*. Bull. Belg. Math. Soc. Simon Stevin **13** (2006), pp. 689–701.

- [Ga-S1] R. Garcia and J. Sotomayor, *Structurally stable configurations of lines of mean curvature and umbilical points on surfaces immersed in \mathbb{R}^3* . Publ. Mat. **45** (2001), pp. 431–466.
- [Ga-S2] R. Garcia and J. Sotomayor, *Lines of Mean Curvature on surfaces immersed in \mathbb{R}^3* , Qualit. Theory of Dyn. Syst., **5**, (2004), pp. 137-183.
- [Ga-S3] R. Garcia and J. Sotomayor, *Differential Equations of Classical Geometry, a Qualitative Theory*, Publicações Matemáticas, 27^o Colóquio Brasileiro de Matemática, IMPA, (2009).
- [Gu] A. Gullstrand, *Zur Kenntiss der Kreispunkte*, Acta Math. **29**, (1905), pp. 59-100,
- [G-S1] C. Gutierrez and J. Sotomayor, *Structural Stable Configurations of Lines of Principal Curvature*, Asterisque **98-99**, (1982), pp. 185-215.
- [G-S2] C. Gutierrez and J. Sotomayor, *Principal lines on surfaces immersed with constant mean curvature*. Trans. Amer. Math. Soc. **293** (1986), pp. 751–766.
- [Ha] J.G. Hardy, *Darboux lines on surfaces*, Amer. Journal of Mathematics, **20**, (1898), pp. 283–292.
- [H-J] U. Hertrich-Jeromin, *Introduction to Möbius Differential Geometry*, London Math. Soc. Lecture Notes, vol **300** Cambridge University Press (2003).
- [Ja] C.G.J. Jacobi. *Note vonder geodätischen Linie auf einem Ellipsoid und den verschiedenen Anwendungen einer merkwürdigen analytischen Substitution*, Crelle J. **19** (1839), pp. 309–313.
- [Kl] W. Klingenberg, *Riemannian geometry*, de Gruyter Studies in Mathematics, Berlin (1982).
- [La-Oh] R. Langevin and J. O’Hara, *Conformal arc-length as $\frac{1}{2}$ dimensional length of the set of osculating circles*, Comment. Math. Helvetici, **85**, (2010), pp. 273–312.
- [La-So] R. Langevin and G. Solanes, *Conformal geometry of curves and the length of canals*, Advances in Geometry (2011). **R: add pages when its appears**
- [La-Wa] R. Langevin and P.G. Walczak, *Conformal geometry of foliations* Geom. Dedicata, **132**, (2008), pp. 135–178.
- [Mo1] G. Monge, *Sur les lignes de courbure de l’ellipsoïde*, Journal de l’École Polytechnique II^{ème} cahier, cours de Floréal an III (around 1795) p. 145.
- [Mo2] G. Monge, *Géométrie Descriptive*, Leçons données aux Écoles Normales, l’an 3 de la République, Éditions Jacques Gabay, (1989) (réédition).
- [Mo3] G. Monge et Hachette. *Application de l’analyse à la géométrie, Première Partie*. (1807) pp 1-57.
- [M-N] E. Musso and L. Nicoldi, *Willmore canal surfaces in Euclidean space*, Rend. Istit. Mat. Univ. Trieste **31**, (1999), pp. 177–202.

- [Pe] A. Pell, *D-lines on Quadrics*, Trans. Amer. Math. Soc. vol **1**, (1900) pp. 315–322.
- [Po] I. R. Porteous, *Geometric Differentiation*, Cambridge Univ. Press, (2001).
- [Ri] M. Ribaucour, *Propriétés de courbes tracées sur les surfaces*, Note Comptes Rendus Acad. Scien. Paris, Tome LXXX, (1875), pp. 642-645.
- [Ro] R. Roussarie, *Modèles locaux de champs et de formes*, Astérisque, **30**, (1975).
- [Sa1] L. A. Santaló, *Curvas extremales de la torsion total y curvas-D*, Publ. Inst. Mat. Univ. Nac. Litoral. (1941), pp. 131–156.
- [Sa2] L. A. Santaló, *Curvas D sobre conos*, Select Works of L.A. Santaló, Springer Verlag, (2009), pp. 317-325.
- [Se] F. Semin, *Darboux lines*, Rev. Fac. Sci. Univ. Istanbul (A) **17**, (1952). pp. 351–383.
- [Sp] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, vol. III, Publish of Perish Berkeley, (1979).
- [St] D. Struik, *Lectures on Classical Differential Geometry*, Addison Wesley, (1950), Reprinted by Dover Collections, (1988).
- [Tho] G. Thomsen, *Über konforme Geometrie. II: Über Kreisscharen und Kurven in der Ebene und über Kugelscharen und Kurven im Raum*, Abhandlungen Hamburg **4**, 117–147 (1925).
- [Tr] M. A. Tresse, *Sur les invariants différentiels d'une surface par rapport aux transformations conformes de l'espace*, Note Comptes Rendus Acad. Sci. Paris, **192**, (1892), pp. 948–950.
- [Ve] E. Vessiot, *Contributions à la géométrie conforme. Cercles et surfaces cerclées*. J. Math. Pures Appliqués **2** (1923), pp. 99-165.

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Ro: coloquei o texto abaixo somente para pensar no assunto. Nao vejo essencial neste momento. Trabalhei na eq. no modelo de forma normal mas nao consegui uma prova direta ainda)

Definition 50. *A closed surface M is called a Willmore surface if $\Delta\mathcal{H} + 2\mathcal{H}(\mathcal{H}^2 - \mathcal{K}) = 0$.*

Proposition 51. *A regular canal Willmore surface, free of umbilical points, is isothermic.*

Proof. See [M-N] and [H-J] □

R: With Pawel, I think to have an interpretation of Willmore canals which reduces the problem to finding a one parameter family of circle such that at the contact point where the circle is of curvature k , the two folds of the envelope have the same curvature $2k$. This uses the three model of special canal: cylinder in $\mathbb{S}^2 \times \mathbb{R}$, cylinder in $\mathbb{R}^2 \times \mathbb{R}$ and cylinder in $\mathbb{H}^2 \times \mathbb{R}$ (see our last version of “special canal surfaces of \mathbb{S}^3 ”).